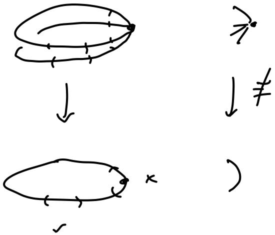
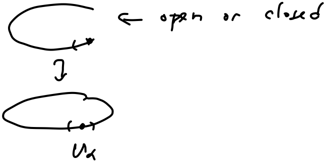
§1.3 Covering spaces (cont)

Ex: X, Y discrete. Any function $f: X \to Y$ is a cover.

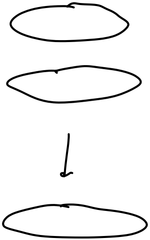
Non-examples:

- 

- 

Ex:
$$\begin{tikzcd} \emptyset \arrow[d] \\ X \end{tikzcd}$$
✓

Ex:



Ex: Any disjoint union of

$$\left\{ \begin{tikzcd} \tilde{X}_\alpha \arrow[d, "p_\alpha"'] \\ X \end{tikzcd} \right\} \rightsquigarrow$$

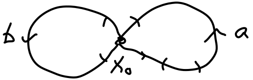
covers ⟹

Exercise 1.3.1: $\tilde{X} \xrightarrow{p} X$ cover, $A \subseteq X$

$$\Rightarrow \quad \tilde{X}|_A := p^{-1}(A) \xrightarrow[p]{} A \quad \text{is a cover.}$$

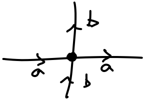
Covers of $S^1 \vee S^1$

$X = S^1 \vee S^1$



Defn: A 2-oriented graph: a graph with:

- edges oriented (directed)
- edges labelled with "a" or "b"
- near each vertex, looks like



$$\begin{tikzcd} S^1 \amalg S^1 \arrow[d, "\nabla"] \\ S^1 \end{tikzcd}$$

Covers of X.

$$\begin{tikzcd} \coprod_\alpha \tilde{X}_\alpha \arrow[d, "p"] \\ X \end{tikzcd}$$

cover

p induced by p_α's.

Any graph $\tilde{X}$ gives a space, also called $\tilde{X}$.
Extra data gives a cts map

$p: \tilde{X} \to X$:

- vertices $\mapsto x_0$
- edges $\mapsto$ left or right loop depending on label.

Fact: This is a cover.

Get every cover of X in this way, up to iso. Iso of 2-oriented graphs iff covers iso.

Note: # of vertices = # a edges
= # b edges

Call this the number of sheets.

Exercise 1.3.10: Find

and 3-sheeted covers of

Choose $\tilde{x}_0 \in p^{-1}(x_0) =$

p induces

$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow$

$\parallel$

free group on $k+1$ gens, where $k = \#$vertices

Fact: p_* is always in

all 2-sheeted

X up to iso.

{vertices}.

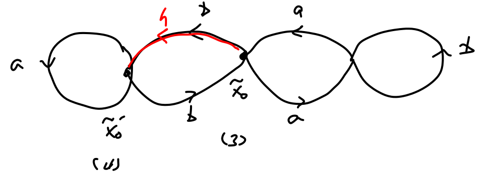
$\pi_1(X, x_0)$

$\parallel$

$\langle a, b \rangle$

njective. $\|$

$\tilde{X}$



$h : \tilde{x}_0 \to \tilde{x}_0'$

$$\beta_h : \pi_1(\tilde{X}, \tilde{x}_0') \xrightarrow{\cong} \pi_1(\tilde{X}, \tilde{x}_0)$$

$$[f] \longmapsto [\underset{\sim}{b} \cdot f \cdot \underset{\sim}{b}^{-1}]$$

So

$$b\, p_*(\pi_1(\tilde{X}, \tilde{x}_0'))\, b^{-1} = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

Main

Fix

and

Then

a

n classification thm 1.38:

path conn, locally path conn, semilocally simply conn (X, x_0).

$$\left\{\begin{matrix}\text{Connected covers}\\ \text{of } X\end{matrix}\right\}\Big/\text{iso} \cong \left\{\begin{matrix}\text{subgps of}\\ \pi_1(X, x_0)\end{matrix}\right\}\Big/\text{conjugacy}$$

$$\tilde{X} \xrightarrow{p} X \longmapsto \text{im}(p_* \text{ for } \tilde{x}_0)$$

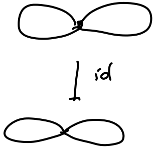
and

$$\left\{\begin{matrix}\text{conn covers of } X\\ \text{with a choice of}\\ \tilde{x}_0\end{matrix}\right\}\Big/\begin{matrix}\text{isos}\\ \text{preserving}\\ \text{basepoint}\end{matrix} \cong \left\{\begin{matrix}\text{subgps of}\\ \pi_1(X, x_0)\end{matrix}\right\}$$

$$(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0) \longmapsto \text{im}(p_*)$$

re bijections

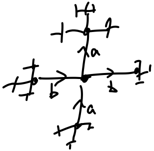
$X = S^1 \vee S^1$



$\text{im}(p_*) =$

What cover corresponds to $0 \subseteq \pi_1(X, x_0)$?

$\tilde{X}$ must be simply conn.



∞-graph with no loops.

"Universal cover".

alld
π,

Lifting

Fix a cover $p: \tilde{X} \to X$ (any X).

Def: A lift of $f: Y \to X$ is $\tilde{f}: Y \to \tilde{X}$ s.t.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes.

Prop 1.30: (Homotopy

htpy $\{f_t : Y \to X\}$

$\exists!$ htpy $\tilde{f}_t$

extended $\tilde{f}_0$.

Pf: The proof of (c

When $Y = pt$, get

$$\begin{tikzcd} & \emptyset \arrow[d] \\ I \arrow[ur, dashed, "\cancel{\beta}"] \arrow[r] & X \end{tikzcd}$$

Recall: If f_t i

then $\underline{\underline{\tilde{f}_t}}$ is a

$s=0$

lifting) Given a

and lift $\tilde{f}_0$ of f_0,

lifting f_t and

) in Thm 1.7 applies. □

path lifting property.

s a htpy of paths,

htpy of paths.

Prop 1.31: Choose $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(\tilde{x})$.

(1) $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

(2) Let f_0 be a loop in X at x_0. Then $[f_0] \in \operatorname{im} p_*$ iff the unique lift $\tilde{f}_0$ starting at $\tilde{x}_0$ is a loop.

Pf: (1) Suppose $[\tilde{f}_0] \in \ker p_*$, so $\tilde{f}_0$ is a loop at $\tilde{x}_0$ and $f_0 := p\tilde{f}_0$ is null homotopic. So $\exists$ htpy f_t of loops with f_1 constant. By htpy lifting, $\exists$ htpy $\tilde{f}_t$ of loops. By uniqueness, f_1 is constant.

(2) $\Leftarrow$: clear.

$\Rightarrow$: $[f_0] \in \operatorname{im}(p_*)$

$\Rightarrow$ $f_0 \simeq f_1$ via f_t, with

$f_1 = p\tilde{f}_1$ for $\tilde{f}_1$ a loop at $\tilde{x}_0$.

By htpy lifting, $\exists$ lifted htpy $\tilde{f}_t$ of loops. So $\tilde{f}_0$ is a loop. $\square$

Def: $|p^{-1}(x)|$ is a locally const. fn on X. So if X is connected, then it is constant. This is called the *number of sheets* of p.

Prop 1.32: Assume X, $\tilde{X}$ path conn, $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(X)$. Let $G := \pi_1(X, x_0)$ and $H := \operatorname{im}(\pi_1(\tilde{X}, \tilde{x}_0)) \leq G$. Then

$\#$ of sheets $= [G : H] = |H \backslash G|$.

Pf: Define

$$\Phi : H\backslash G \longrightarrow p^{-1}(x_0$$

$$H[g] \longmapsto \tilde{g}(1)$$

where g is a loop in X and $\tilde{g}$ is the lift starting

Well-defined:

- $\tilde{g}(1)$ depends only on since htpy lifts to *of paths*.
- $\widetilde{h\cdot g} = \tilde{h}\cdot\tilde{g}$, since loop at $\tilde{x}_0$, by 1.31 $([h] \in H)$.

So $\widetilde{h\cdot g}(1) = \tilde{g}($

)

at x_0

at $\tilde{x}_0$.

$[g]$

htpy

$\tilde{h}$ is a

(2).

(1).

Surjective: Given $\tilde{x} \in p^{-1}(x_0)$.

Choose path $\tilde{g}$ from $\tilde{x}_0$ to $\tilde{x}$.

Then $g := p\tilde{g}$ is a loop at x_0.

And $\Phi(H[g]) = \tilde{g}(1) = \tilde{x}_0$.

injective: $\Phi(H[g_1]) = \Phi(H[g_2])$

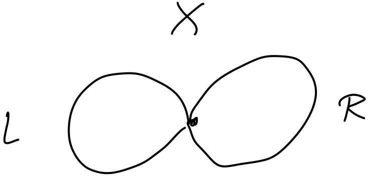
$\Rightarrow \tilde{g}_1(1) = \tilde{g}_2(1)$

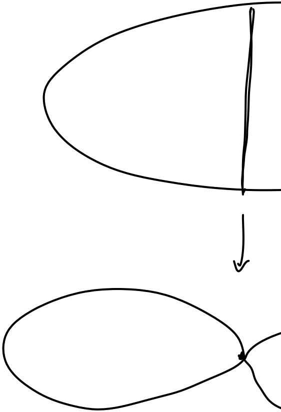
$\Rightarrow g_1 \cdot \bar{g}_2$ lifts to a loop

$(\tilde{g}_1 \cdot \bar{\tilde{g}}_2)$

$\Rightarrow [g_1][g_2]^{-1} \in H$

$\Rightarrow H[g_1] = H[g_2]$. $\square$





Suppose we have a def. retraction H from X onto L.

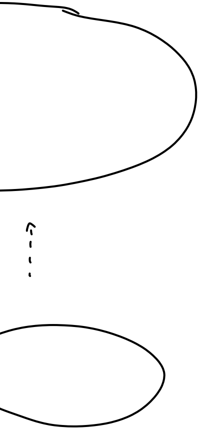
Write r for the retraction onto R

$$r|_R = id_R$$

$$r|_L = \text{constant at } x_0.$$

Then $rH|_R$ is a def. retraction from R to x_0.

π_1 $\mathbb{Z}$ π_1 triv.



Lecture 10

2022/2/10

Recall:

Prop 1.32: Assume X, $\tilde{X}$ path conn, $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(X)$. Let $G := \pi_1(X, x_0)$ and $H := \mathrm{im}(\pi_1(\tilde{X}, \tilde{x}_0)) \le G$. Then

$$\# \text{ of sheets} = [G : H] = |H \backslash G| .$$

Ex: $\mathbb{RP}^2 := S^2/\sim$, $x \sim -x$.

$S^2 \xrightarrow{q} \mathbb{RP}^2$ is a cover.

2-sheeted.

$\pi_1(S^2) = 0$, so $H = 0$.

By 1.32, $|\pi_1(\mathbb{RP}^2)| = 2$.

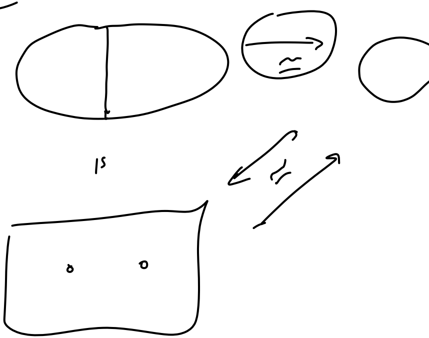
So $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2$.

$(S^1, S^2, S^1 \vee S^1, \mathbb{RP}^2)$

So $\exists$ a loop γ in $\mathbb{RP}^2$ s

$\gamma \not\simeq$ const but $\gamma^2 \simeq$ const.

HW2



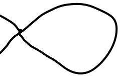
Def: Y is loc. path conn

if $\forall y \in Y$ $\forall$ nbhd U of y $\exists$

$y \in V \subseteq U$ s.t. V is

connected.

-t.



ected

open

path

Prop 1.33 (Lifting criterion): Let Y be path conn + loc. path conn. Let $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ be a cover. Let $y_0 \in Y$ and $f : (Y, y_0) \to (X, x_0)$.

A lift $\tilde{f}$ exists

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0)
\end{tikzcd}$$

iff $f_* \pi_1(Y, y_0) \subseteq p_* \pi_1(\tilde{X}, \tilde{x}_0)$.

Note: Answer depends on $\tilde{x}_0$. If you don't care about basepoints, need to check $f_* \pi_1(Y, y_0) \subseteq$ some conjugate of $p_* \pi_1(\tilde{x}, \tilde{x}_0)$.

Prop 1.3

of Y

at one

Note: So

Pf sketch

is

Since Y

Prop 1.37:

$p_1 : (\tilde{X}_1,$

path conn

$\exists$ no

: Y connected, $\tilde f_1, \tilde f_2$ lifts

$\xrightarrow{f} X$. If $\tilde f_1$ and $\tilde f_2$ agree

point, then $\tilde f_1 = \tilde f_2$.

$\tilde f$ in 1.33 is unique.

: $\{ y \in Y \mid \tilde f_1(y) = \tilde f_2(y) \}$

open, closed and non-empty.

connected, it is all of Y. □

X path conn, loc. path conn.

$x_1) \to (X, x_0)$, $p_2 : (\tilde X_2, \tilde x_2) \to (X, x_0)$

covers. Then

$$\begin{tikzcd}
(\tilde X_1, \tilde x_1) \arrow[r, phantom, "\overset{\Phi}{\cong}"] \arrow[dr, "p_1"'] & (\tilde X_2, \tilde x_2) \arrow[d, "p_2"] \\
 & (X, x_0)
\end{tikzcd}$$

iff

$$(p_1)_* \pi_1(\tilde X_1, \tilde x_1) = (p_2)_* \pi_1(\tilde X_2, \tilde x_2).$$

Pf: $\Rightarrow$: clear.

$\Leftarrow$: By 1.33, get ptd lifts

$$\begin{tikzcd}
\tilde X_1 \arrow[r, dashed, "\tilde p_1"] \arrow[dr, "p_1"'] & \tilde X_2 \arrow[r, dashed, "\tilde p_2"] \arrow[d, "p_2"] & \tilde X_1 \arrow[dl, "p_1"] \\
 & X &
\end{tikzcd}$$

So $\tilde p_2 \tilde p_1$ is a lift of p_1 through p_1 sending $\tilde x_1$ to $\tilde x_1$. By 1.34, $\tilde p_2 \tilde p_1 = id_{\tilde X_1}$. Same with other composite. So $\Phi = \tilde p_1$ is a homeomorphism. □

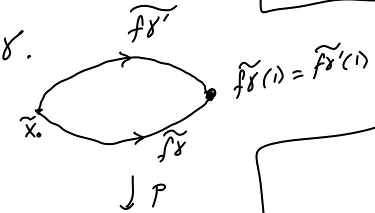
Pf of 1.33: $\Rightarrow$: clear.

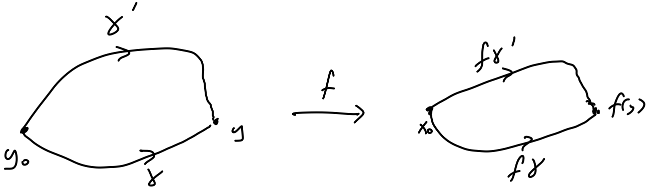
$\Leftarrow$: Let $y \in Y$. Choose path γ from y_0 to y. So $f\gamma$ is a path from x_0 to $f(y)$ in X. It has a lift $\widetilde{f\gamma}$ starting at $\tilde{x}_0$.

Define $\tilde{f}(y) = \widetilde{f\gamma}(1)$.

Then $p\tilde{f}(y) = p\,\widetilde{f\gamma}(1) = f\gamma(1) = f(y)$.

Claim: $\tilde{f}(y)$ indep. of γ.





$\gamma \cdot \bar{\gamma}'$ is a loop at y_0

$f\gamma \cdot \overline{f\gamma'}$

Loop $f\gamma \cdot \overline{f\gamma'}$

By 1.31(2),

So must have

$\tilde{f}$ cts: See

Ex: 1.3.9.

$\in \operatorname{im} f_* \subseteq \operatorname{im} p_*$.

, it lifts to a loop in $\tilde{X}$.

$\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$.

text. $\square$

Universal covers

A cover $\tilde{X} \to X$ with $\tilde{X}$ simply conn

is called a *universal cover*.

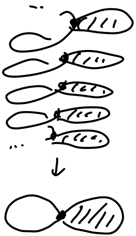
Ex: $\mathbb{R} \xrightarrow{p} S^1$.

Ex: $\mathbb{R} \times \mathbb{R} \xrightarrow{p \times p} S^1 \times S^1$

Ex: $S^2 \xrightarrow{id} S^2$

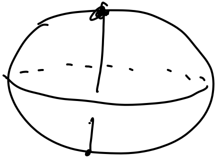
Ex: $S^2 \xrightarrow{q} \mathbb{R}P^2$

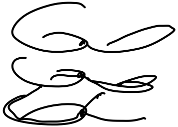
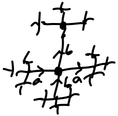
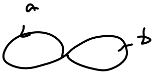
Ex:



$S^1 \vee D^2$

Ex 1.3.4 : Find universal covers of



and $\mathbb{RP}^2 \vee S^1$.