Lecture 3

2022/1/18

- HW1 due Saturday night
- Read §1.1.

§1.1 The fundamental group

Def: A path from x_0 to x_1 in X is a map $f: I \to X$ s.t. $f(0) = x_0$ and $f(1) = x_1$.

A homotopy of paths is a homotopy $\{f_t : I \to X\}_{t \in I}$ s.t.

$$(*) \quad \begin{array}{l} f_t(0) = x_0 \\ f_t(1) = x_1 \end{array} \quad \forall t \in I.$$

Say f_0, f_1 are

Write $f_0 \simeq f_1$.

Note: Paths and ho

self-intersections :

Ex 1.1: Linear ho

$f_0, f_1 : I \to \mathbb{R}^n$

and $f_0(1) = f_1(1)$.

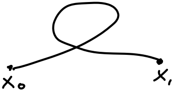
Then they are homotopic via

$$f_t(s) = (1-$$

Check: cts, $(*)$

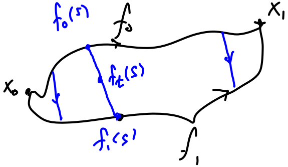
homotopic .

motopies can have



motopies : Let

with $f_0(0) = f_1(0)$



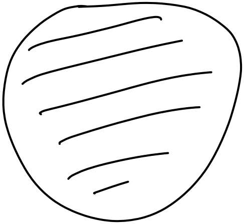
$t) f_0(s) + t f_1(s)$

.

True for paths in any convex subspace X of $\mathbb{R}^n$: $x, y \in X$, $t \in I$

$\Rightarrow (1-t)x + ty \in X$.

$D^2 =$



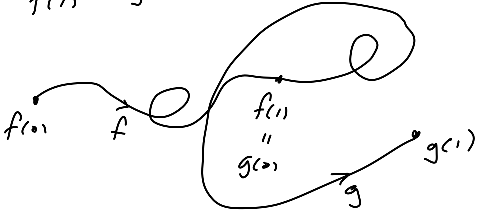
Prop 1.2: Fix $x_0, x_1 \in X$. The relation $\simeq$ on the paths from x_0 to x_1 is an equivalence relation.

Pf: Same proof as last time. Just check (*). $\square$

Write $[f]$ for the equivalence class.

Composition: Given $f, g : I \to X$

with $f(1) = g(0)$.



Define $f \cdot g : I \longrightarrow X$ by

$$(f \cdot g)(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Cts by that lemma.

Lemma: If $f_0 \simeq$

$g_0 \simeq g_1$ via $\{g_t$

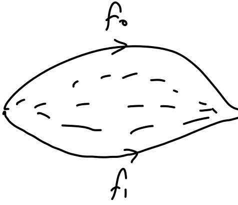
then

$$f_0 \cdot g_0 \simeq$$

via $f_t \cdot g_t$.

Pf: Cts by tha

$(*)$ easy.



f_1 via $\{f_t\}$ and

$\}$, with $f_0(1) = g_0(0)$,

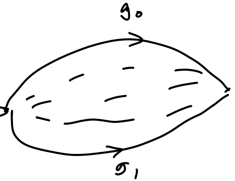
$$\begin{matrix} f_0(1) & = & g_0(0) \\ \shortparallel & & \shortparallel \\ f_1(1) & & g_1(0) \end{matrix}$$

$f_1 \cdot g_1$?

← well-defined by $(*)$

t lemma.

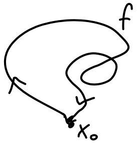
□



$\pi_1(X, x_0)$

Fix a base point $x_0 \in X$.

A path in X from x_0 to x_0 is a loop (based) at x_0.



Define $\pi_1(X, x_0) = \{[f] \mid f$ is a loop at $x_0\}$.

Ex 1.4: X convex $\Rightarrow \pi_1(X, x_0)$ has one element.

Prop 1.3: $\pi_1(X, x_0)$ is a group under $[f][g] = [f \cdot g]$.

Pf: Product well-defined:

- $f(1) = x_0 = g(0)$, so $f \cdot g$ makes sense
- $[f \cdot g]$ depends only $[f]$ and $[g]$ by Lemma

Reparametrization Lemma: Given a path f and any $\Phi : I \to I$ with $\Phi(0) = 0$ and $\Phi(1) = 1$, we have $f \simeq f \circ \Phi$ as paths.

Proof: Use htpy $f \circ \Phi_t$ where

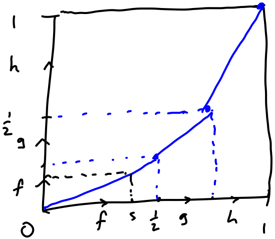
$$\Phi_t(s) = (1-t)\,\Phi(s) + ts.$$

$\Phi_1 = \mathrm{id}_I$, $\Phi_0 = \Phi$, so

$f \circ \Phi_1 = f$ and $f \circ \Phi_0 = f \circ \Phi$. $\square$

Associativity: Given paths in X with $f(1) = g(0)$ and then $f \cdot (g \cdot h)$ is a repara of $(f \cdot g) \cdot h$ by

$$\Phi(s) = \begin{cases} \frac{1}{2}s, & 0 \\ s - \frac{1}{4}, & \frac{1}{2} \\ 2s - 1, & 3/4 \end{cases}$$



$$f \cdot (g \cdot h) = \underline{((f \cdot g) \cdot h) \circ \Phi}$$

f, g, h

$g(1) = h(0)$,

matrization

$\leq s \leq \frac{1}{2}$

$\leq s \leq 3/4$

$\leq s \leq 1$.

✓

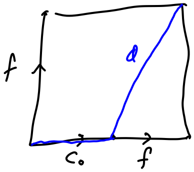
.

So, for loops f, g, h,

$$[f]([g][h]) = ([f][g])[h] .$$

Identity: Let $f: I \to X$ be a path from x_0 to x_1. Let c_0 be the constant path at x_0. Then $c_0 \cdot f$ is a reparamatrization of f by

$$\Phi(s) = \begin{cases} 0, & 0 \leq s \leq \frac{1}{2} \\ 2s-1, & \frac{1}{2} \leq s \leq 1 . \end{cases}$$



$$c_0 \cdot f = f \circ \Phi .$$

Similarly fo

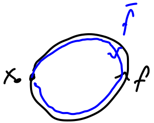
So, for loops

$[c_0][f] =$

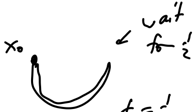
Inverses: Given

$\bar{f}(s) = f(1-$

$f \cdot \bar{f} \simeq c_0$:



$t = 0$



$t = \frac{1}{2}$

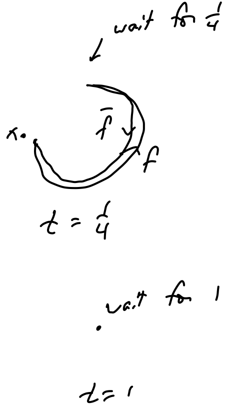
c_1, constant at x_1.

, set

. $[f] = [f][c_0]$.

a loop f, let

s). Then



Exercise: Give formula.

$\bar{f} \cdot f \simeq c_0$ similar.

So $[f]^{-1} = [\bar{f}]$. $\cancel{f^{-1}}$ $\square$

Inverses works for paths as well.

Exercise 1.1.1: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$, then $f_0 \simeq f_1$.

(All paths.)

Induced homomorphisms

Consider $\phi : X \to Y$ with $\phi(x_0) = y_0$. Write $\phi : (X, x_0) \to (Y, y_0)$.

Then get homomorphism

$$\phi_* = \pi_1(\phi) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

$$[f] \longmapsto [\phi \circ f]$$

$$\begin{tikzcd} I \arrow[r, "f"] & X \arrow[r, "\phi"] & Y \\ 0,1 \arrow[r, mapsto] & x_0 \arrow[r, mapsto] & y_0 \end{tikzcd}$$

- $\phi \circ f$ *is* a loop at y_0.
- If $\{f_t\}$ is a homotopy, then $\{\phi \circ f_t\}$ is a homotopy, so ϕ_* is well-defined.
- $\phi \circ (f \cdot g) = (\phi \circ f) \cdot (\phi \circ g$

It is also a *functor*:

- For $(X, x_0) \xrightarrow{\phi} (Y, y_0)$

  $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.
- $(id_X)_* = id_{\pi_1(X, x_0)}$.

Cor: If ϕ is a homeo

then ϕ_* is an isomorphi

Pf: Let ψ be the inverse

Then $\phi_* \circ \psi_* = (\phi \circ \psi)_* =$

And $\psi_* \circ \phi_* = \cdots$

)

$\xrightarrow{\psi} (Z, z_0)$,

morphism,

m.

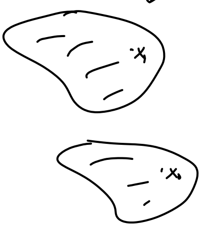
of θ.

$(id)_* = id$.

$= id$.

$\square$

Dependence on x_0



If Y is the path component of X containing x_0, then

$$\pi_1(X, x_0) \cong \pi_1(Y, x_0).$$

If x_0, x_1 are in different path components of X, then there is no relationship between

$$\pi_1(X, x_0) \text{ and } \pi_1(X, x_1).$$

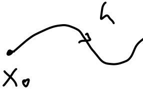
So suppose x_0, x_1 are in the same path component of X.

$\exists$ path $h : I \to X$ from x_0 to x_1.

Define

$\beta_h : \pi_1($

by $\beta_h([f])$



Prop 1.5: β_h

Pf: well-defined

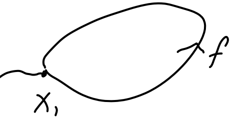
$h \cdot f_0 \cdot \bar{h} \simeq$

homomorphism:

$= \beta_h([$

$, x_1) \to \pi_1(X, x_0)$

$= [\, h \cdot f \cdot \bar{h} \,]$



is a group iso.

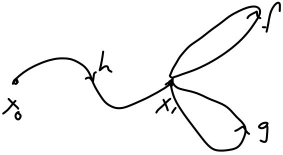
: $f_0 \simeq f_1 \implies$

$h \cdot f_1 \cdot \bar{h}$.

$\beta_h([f][g])$

$f \cdot g])$

$= [h \cdot f \cdot g \cdot \bar{h}]$



$= [h \cdot f \cdot \underline{\bar{h} \cdot h} \cdot g \cdot \bar{h}]$

$\bar{h} \cdot h \simeq c_{x_1}$

$= [h \cdot f \cdot \bar{h}][h \cdot g \cdot \bar{h}]$

$= \beta_h([f])\,\beta_h([g])$.

iso: The map $\beta_{\bar{h}}$ is the inverse:

$$\beta_h \beta_{\bar{h}}[f] = \beta_h[\bar{h} \cdot f \cdot h]$$

$$= [\underline{h \cdot \bar{h}} \cdot f \cdot \underline{h \cdot \bar{h}}]$$

$$= [f]$$

and same for other order. □

Note: Iso depends on h.

Exercises: 1.1.3, 1.1.15.

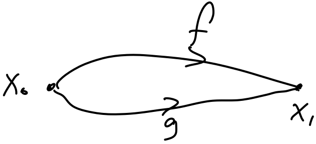
Simple-connectedness

Def: X is simply connected if it is path connected and has trivial π_1.

Prop 1.6: X is simply connected iff $\forall\, x_0, x_1 \in X$ $\exists!$ homotopy class of paths from x_0 to x_1.

Proof: $\Rightarrow$: Assume X simply connected. Let $x_0, x_1 \in X$. Since path conn, $\exists$ path f from x_0 to x_1. Suppose g is another path from x_0 to x_1

$$f \simeq f \cdot \underbrace{\bar{g} \cdot g}_{\simeq c_{x_1}} \simeq$$



$\Leftarrow$: Path conn tr

Also, all paths

are homotopic,

is trivial.

. Then

$c_{x_0} \cdot g \simeq g$

$\uparrow$ since $\pi_1(X, x_0)$ is trivial.

ivial.

x_0 to x_0

i.e. $\pi_1(X, x_0)$

$\square$

$\pi_1(S^1) \cong \mathbb{Z}$

$$S^1 = \{ x \in \mathbb{R}^2 \mid \|x\| = 1 \}$$

$x_0 = (1, 0)$.

Given $n \in \mathbb{Z}$, define $\omega_n : I \to S^1$ by

$$\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s).$$

This is a loop at x_0.

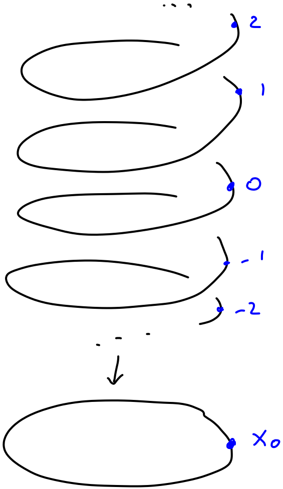
Thm 1.7:

$$\begin{aligned} \Phi : \mathbb{Z} &\longrightarrow \pi_1(S^1, x_0) \\ n &\longmapsto [\omega_n] \end{aligned}$$

is a group iso.

Pf:

$$\begin{tikzcd}
\mathbb{R} \arrow[d, "p"'] & s \arrow[d, mapsto] \\
S^1 & (\cos 2\pi s, \sin 2\pi s)
\end{tikzcd}$$



Let $\tilde{\omega}_n : I \to \mathbb{R}$ be

$$\tilde{\omega}_n(s) = ns.$$

Path from 0 to n.

$$\omega_n = p \circ \tilde{\omega}_n$$

We say $\tilde{\omega}_n$ is a lift of ω_n.

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Pf of Thm 1.7 (cont):

Note: If $\tilde{f} : I \to \mathbb{R}$ is any path from 0 to n, then $\tilde{f} \simeq \tilde{\omega}_n$, so

$$\Phi(n) = [\omega_n] = [p\tilde{\omega}_n] = [p\tilde{f}].$$

Φ is a homomorphism: Let

$$\tau_m : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x+m.$$

Then

$$\Phi(m+n) = [p(\tilde{\omega}_m \cdot \tau_m\tilde{\omega}_n)]$$

(by Note)

$$= [(p\tilde{\omega}_m)\cdot(p\tau_m\tilde{\omega}_n)]$$

$$= [p\tilde{\omega}_m][p\tau_m\tilde{\omega}_n]$$

$$= [\omega_m][p\tilde{\omega}_n] \qquad (p\tau_m$$

$$= [\omega_m][\omega_n]$$

$$= \Phi(m)\Phi(n).$$

Φ is an iso: Use:

(a) $\forall f : I \to S^1$ start

and $\forall \tilde{x}_0 \in p^{-1}(x_0)$,

$\tilde{f}$ of f starting

(b) $\forall$ htpy $f_t : I \to S$

starting at x_0 and

$\tilde{x}_0 \in p^{-1}(x_0)$, $\exists!$ lif

$\tilde{f}_t$ of **paths** starting

$= p$)

ins at x_0

$\exists!$ lift

at $\tilde{x}_0$.

' of paths

each

ted htpy

at $\tilde{x}_0$.

Φ is surjective:

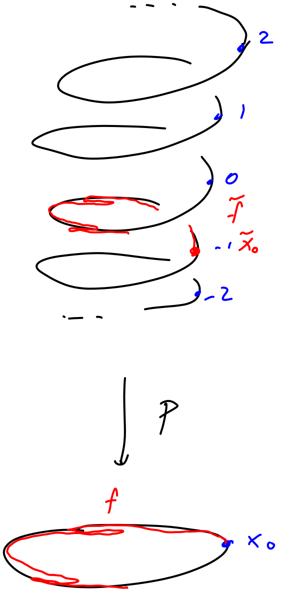
Let $f: I \to S^1$ be a loop at x_0. By (a), there is a lift $\tilde{f}$ starting at 0. Then $p\tilde{f}(1) = f(1) = x_0$, so $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$. By note, $\Phi(n) = [p\tilde{f}] = [f]$.

Φ is injective: Suppose $\Phi(m) = \Phi(n)$. So $\omega_m \simeq \omega_n$. Let f_t be a htpy from $f_0 = \omega_m$ to $f_1 = \omega_n$. By (b), f_t can lift to a htpy $\tilde{f}_t$ of paths starting at 0. By uniqueness in (a), $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. So

$$m = \tilde{\omega}_m(1) = \tilde{f}_0(1) = \tilde{f}_1(1) = \tilde{\omega}_n(1) = n.$$

$\mathbb{R}$

S^1



Need to prove:

(a) $\forall f : I \to S^1$ starting at x_0 and $\forall \tilde{x}_0 \in p^{-1}(x_0)$, $\exists!$ lift $\tilde{f}$ of f starting at $\tilde{x}_0$.

(b) $\forall$ htpy $f_t : I \to S^1$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists!$ lifted htpy $\tilde{f}_t$ of paths starting at $\tilde{x}_0$.

Prove them using:

(c) For any space Y, $\forall F : Y \times I \to S^1$, $\forall \tilde{G} : Y \times \{0\} \to \mathbb{R}$ lifting $F|_{Y \times \{0\}}$, $\exists! \tilde{F} : Y \times I \to \mathbb{R}$ lifting F and extends $\tilde{G}$:

$$\begin{tikzcd}
Y\times\{0\} \arrow[r, "\widetilde{G}"] \arrow[d, hook] & \mathbb{R} \arrow[d, "p"] \\
Y\times I \arrow[r, "F"'] \arrow[ur, dotted, "\widetilde{F}"] & S^1
\end{tikzcd}$$

(a) is the special case of (c) where Y has one point.

Proof of (b) from (c): We are given $F : I\times I \to S^1$ with $F(s,t) = f_t(s)$.
By (a), $\exists!$ lift $\tilde{f}_0$ of f_0 starting at $\tilde{x}_0$.
So take $Y = I$.
Let $\widetilde{G}(s,0) = \tilde{f}_0(s)$.
By (c), $\exists!$ extension/lift $\widetilde{F} : I\times I \to \mathbb{R}$.

Must show $\widetilde{F}$ is

Since $\widetilde{F}(0,t)$ li

at x_0: $p\widetilde{F}(0,t)$

Also, $\widetilde{F}(0,0) = \tilde{x}_0$.

$\widetilde{F}(0,t) = \tilde{x}_0$.

Similarly, $\widetilde{F}(1,t$

So $\widetilde{F}(1,t)$ is a

Prove (c) using

$(*)$ $\exists$ an open co

s.t. each F

disjoint union

each mapped h

~~onto~~ U_α.

a htpy of paths.

ifts constant path

$= F(0,t) = f_t(0) = x_0$.

By uniqueness in (a),

, lifts a constant path,

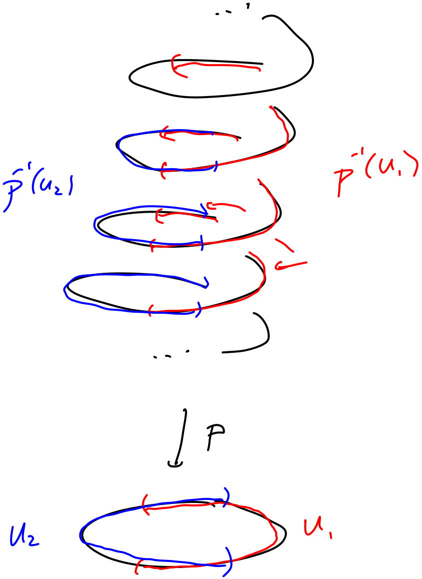
constant path. ∎ (b)

ven $\{U_\alpha\}$ of S^1

$s^{-1}(U_\alpha)$ is a

of open sets

omeomorphically



Easy to verify $(*)$.

Lecture 5

2022/1/25

- HW1 will be returned soon.
- HW2 will be assigned in a few days, due Saturday, Feb 5.
- Questions?

Thm 1.7: $\pi_1(S^1, x_0) \cong \mathbb{Z}$

We need to show (*) $\Rightarrow$ (c).

Recall: $p : \mathbb{R} \to S^1$

$s \mapsto (\cos 2\pi s, \sin 2\pi s)$

$x_0 = (1, 0) \in S^1$.

(*) $\exists$ an open cover

S^1 s.t. each $p^{-1}(U_\alpha)$

disjoint union of open

each of which is mapped

morphically onto U_α by

(c) For every space Y,

$F : Y \times I \to S^1$ and $\tilde{G}$

lifting $F|_{Y\times\{0\}}$, $\exists!\ \tilde{F}$

lifting F and extending

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\tilde{G}"] \arrow[d, hook] & \mathbb{R} \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dotted, "\tilde{F}" description] & S^1
\end{tikzcd}$$

✓ ✓

$\{U_\alpha\}$ of

is a

sets

homeo-

p.

$: Y \times \{0\} \to \mathbb{R}$

$: Y \times I \to \mathbb{R}$

$\tilde{G}$:

.

Pf of (c):

Fix $y_0 \in Y$. For each $t \in I$, can choose nbhd $N_t \times I_t$ of (y_0, t) s.t. $F(N_t \times I_t) \subseteq U_\alpha$, for some α.

$y_0 \in N_t \underset{\text{open}}{\subseteq} Y$, $t \in I_t \underset{\text{open}}{\subseteq} I$.

I compact $\Rightarrow$ finitely many I_t cover I.

Let $N = \cap N_t$ for those t's.

So $y_0 \in N \underset{\text{open}}{\subseteq} Y$.

$\exists\ 0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ s.t. $F(N \times [t_i, t_{i+1}]) \subseteq$ some U_α call it U_i.

Define $\tilde{F}$ on $N \times \{0\}$ as restriction of $\tilde{G}$.

Assume $\tilde{F}$ defi

lifting $F|_{N \times [0, t_i]}$

$\exists$ open $\tilde{U}_i \subseteq$

$p: \tilde{U}_i \cong U_i$

Replacing N with

if necessary, we

Define $\tilde{F}$ on N

F composed with

Repeat to get

lifting $F|_{N \times I}$

ined on $N \times [0, t_i]$,

.

$\mathbb{R}$ s.t.

and $\tilde{F}(y_0, t_i) \in \tilde{U}_i$.

a smaller open nbhd

can assume $\tilde{F}(N \times \{t_i\}) \subseteq \tilde{U}_i$.

$\times [t_i, t_{i+1}]$ to be

h $p^{-1} : U_i \to \tilde{U}_i$.

$\tilde{F} : N \times I \to \mathbb{R}$

and extends $\tilde{G}|_{N \times \{0\}}$.

Uniqueness in case $Y = \{*\}$.

Suppose $\tilde{F}, \tilde{F}' : I \to \mathbb{R}$ are two lifts of $F : I \to S^1$ s.t. $\tilde{F}(0) = \tilde{F}'(0)$.

Choose $0 = t_0 < t_1 < \dots < t_m = 1$ s.t. $F([t_i, t_{i+1}]) \subseteq$ some U_i.

Assume $\tilde{F} = \tilde{F}'$ on $[0, t_i]$.

$[t_i, t_{i+1}]$ connected $\Rightarrow \tilde{F}([t_i, t_{i+1}])$ lands in a single $\tilde{U}_i$.

$\tilde{F}'([t_i, t_{i+1}])$ land in same $\tilde{U}_i$, since $\tilde{F}(t_i) = \tilde{F}'(t_i)$.

Since $p|_{\tilde{U}_i}$ is injective and $p\tilde{F} = F = p\tilde{F}'$, so $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$.

By induction, $\tilde{F} = \tilde{F}'$ on I.

Note: Uniqueness for any Y follows, by restricting to $\{y_0\}\times I$ for every $y_0 \in Y$.

Existence: We've constructed $\tilde{F} : N \times I \to \mathbb{R}$ for nbhds N of each $y_0 \in Y$. By uniqueness, these all agree on sets of form $\{y\} \times I$. So get a well-defined fn $\tilde{F} : Y \times I \to \mathbb{R}$. Its continuous, since cts on an open cover. □

Q: Does every map $Y \to S^1$ have a lift? No.

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d, "p"] \\
S^1 \arrow[r, "id"'] \arrow[ur, dashed, "\cancel{f}"] & S^1 .
\end{tikzcd}$$

Consequences of $\pi_1(S^1) \cong \mathbb{Z}$

Cor: ∄ retraction $D^2 \to S^1$.

Pf: As in 1st lecture, with basepoints added. □

Thm 1.9: Brouwer fixed pt thm.

Exercise 1.1.16 – do it.

Thm 1.10 (Borsuk-Ulam Thm):

For every $f: S^2 \to \mathbb{R}^2$, $\exists\, x \in S^2$ s.t. $f(x) = f(-x)$.

Pf: Suppose false for $f: S^2 \to \mathbb{R}^2$.

Define $g: S^2 \to S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$ ← never zero

✓ Define loop η in S^2 by

$$\eta(s) = (\cos 2\pi s, \sin 2\pi s, 0)$$

Let $h = g\eta$, a loop in S^1.

Since $g(-x) = -g(x)$,

$h(s + \frac{1}{2}) = -h(s)$

Let $\tilde{h} : I \to \mathbb{R}$ be

$\tilde{h}(s + \frac{1}{2}) = \tilde{h}(s) +$

$n \in \mathbb{Z}$. By conti

indep. of s. So

$\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + n$

$= \tilde{h}(0) + 2n$

Since $2n+1 \neq 0$,

in $\pi_1(S^1, x_0)$.

But η is null

we have

for $s \in [0, \frac{1}{2}]$.

a lift.

$n + \frac{1}{2}$ for

inuity, n is

$+ \frac{1}{2}$

$+ 1$

so $[h] \neq 0$

homotopic.

⨳. □

Exercise 1.1.8: $S^1 \times S^1 \longrightarrow \mathbb{R}^2$?

$$\begin{array}{c} (x,y) \\ \text{I} \\ (-x,-y) \end{array}$$

Exercise 1.1.9: Ham-Sandwich thm.

$A_1, A_2, A_3 \subseteq \mathbb{R}^3$

$\exists$ plane P that cuts all three in half.

Thm 1.8: Every non-constant poly with coefts in $\mathbb{C}$ has a root in $\mathbb{C}$.

Pf: Let $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$

Assume no roots. For $r \geq 0$, define $f_r : I \to S^1$ by

$$f_r(s) = \phi\left(p(re^{2\pi i s})/p(r)\right)$$

where $\phi : \mathbb{C} \setminus \{0\} \longrightarrow S^1$

$$z \longmapsto \frac{z}{|z|}.$$

$f_r(0) = f_r(1) = 1 \in S^1 \subseteq \mathbb{C}$.

For $r = 0$, f_0 is const.

Choose r_0 large enough s

that $|z^n| > |a_1 z^{n-1} + \dots + a_n$

when $|z| = r_0$.

Let $p_t(z) = z^n + t(a_1 z^{n-1} +$

Note p_t has no roots with

for $0 \leq t \leq 1$.

Replace p by p_t in formul

for f_{r_0}.

As t goes from 1 to 0,

get a htpy from f_{r_0} to

$w_n : w_n(s) = e^{2\pi i n s}$.

0

1

$\cdots + a_n)$.

$|z| = r_0$

a

Compose htpies to get

$$\text{const} = f_0 \simeq f_{r_0} \simeq w_n$$

$r=0$ ---- $r=r_0$ -- $r=r_0$

$t=1$ --- $t=1$ -- $t=0$

So $0 = n$. ※ since $n > 0$. $\square$

Other calculations

Prop 1.12: $x_0 \in X$, $y_0 \in Y$.

$$\pi_1(X\times Y, (x_0,y_0)) \cong \pi_1(X,x_0)\times \pi_1(Y,y_0).$$

Idea: A function $Z \xrightarrow{f} X\times Y$ is the same as 2 functions $Z \xrightarrow[g]{} X$, $Z \xrightarrow[h]{} Y$.

f cts iff

Homotopies

Composition

Ex 1.13: To

$\pi_1(S^1\times S^1,$

(w_p, w_q)

Exercise 1.1

g and h cts.

agree too.

matches. □

rus $S^1 \times S^1$!

$(x_0, x_0)) \cong \mathbb{Z} \times \mathbb{Z}$

$\cong \mathbb{Z} \oplus \mathbb{Z}$.

$\longleftarrow\!\shortmid \ (p, q)$

10.

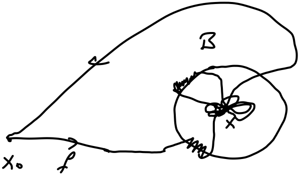
Prop 1.14: $\pi_1(S^n) = 0$ if $n \geq 2$.

Pf: Let f be a loop in S^n at some $x_0 \in S^\circ$.

If $\exists\, x \in S^n$ not in image of f, then f lands in $S^n \setminus \{x\} \cong \mathbb{R}^n$, which is simply connected, so f is htpic to a constant.

So suffices to show f is htpic to a loop that misses a chosen point $x \neq x_0$.

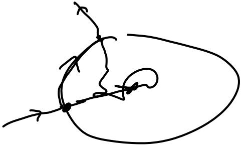
Choose a small open ball $B \ni x$ ($x_0 \notin B$).



f can enter B, hit x and leave B only finitely many times.

[$f^{-1}(B)$ is an open cover of $\underset{\overset{\uparrow}{\text{compact.}}}{f^{-1}(x)}$]

For each of those, just homotope the portion of f inside B to the boundary of B.



New path misses x.

□

Ex 1.15: $\mathbb{R}^n - \{x\} \cong$

So $\pi_1(\mathbb{R}^n - \{x\}) \cong$

$\cong \pi$

$\cong \{$

Cor 1.16: $\mathbb{R}^2 \not\cong \mathbb{R}^n$

Pf: Let $f : \mathbb{R}^2 \cong \mathbb{R}^n$

$n = 0$: clear.

$n = 1$: $\underset{\text{path conn}}{\mathbb{R}^2 - \{0\}} \cong \mathbb{R}^1$

$n > 2$: $\pi_1(\mathbb{R}^2 - \{0\})$

$\pi_1(\mathbb{R}^n - \{0\})$

$S^{n-1} \times \mathbb{R}$

$\pi_1(S^{n-1} \times \mathbb{R})$

$\pi_1(S^{n-1}) \times \underset{0}{\underline{\pi_1(\mathbb{R})}}$

$\mathbb{Z}, \quad n=2$

$0, \quad \text{else}.$

for $n \neq 2$.

.

$\#$

$- \{f(0)\}$

not. $\#$

$= \mathbb{Z}$

$\#$

$= 0. \quad \square$

Lecture 6

2022/1/27

- Next week: in person! MC108.

Homotopy equivalences

Fix $A \subseteq X$, $B \subseteq Y$. A *map of pairs* $(X, A) \to (Y, B)$ is a map $X \xrightarrow{f} Y$ s.t. $f(A) \subseteq B$.

A *homotopy of maps of pairs* is a homotopy $\{f_t\}$ s.t. each $f_t : (X, A) \to (Y, B)$. Write $\simeq$.

Ex: $\pi_1(X, x_0) = \{ (I, \{0,1\}) \to (X, x_0) \} / \simeq$

Common case: $(X,$

$f(x_0) = y_0$. "Bas

Def: $f : (X, x_0) \longrightarrow$

base point preserving

if $\exists\, g : (Y, y_0) \longrightarrow ($

$fg \underset{\text{b.p.p.}}{\simeq} id_{(Y, y_0)}$ and

write $(X, x_0) \simeq (Y, y_0)$.

Note: $(X, x_0) \simeq ($

But the converse ca

can choose x_0, y_0..

Lemma: $f_0, f_1 : (X,$

Then $(f_0)_* = (f_1)_* :$

$x_0) \xrightarrow{f} (Y, y_0)$.

e point preserving maps."

(Y, y_0) is a

(b.p.p.) homotopy equiv

(X, x_0) with

$$gf \simeq id_{(X, x_0)}.$$

$Y, y_0) \Rightarrow X \simeq Y$.

fail, even if you

$x_0) \to (Y, y_0)$, $f_0 \simeq f_1$.

$\pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Pf: Let g be a loop in X at x_0.

Then $$(f_0)_*[g] = [f_0 g] = [f_1 g] = (f_1)_*[g]. \quad \square$$
$$(f_0 g \simeq f_1 g)$$

Cor: If f is a b.p.p. ho. equiv, then f_* is iso. $\square$

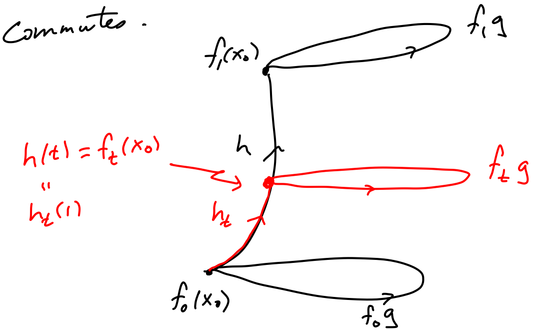
Prop 1.18: If f is a (free) is a ho. equiv, then
$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$
is an iso. $\forall x_0$.

Lemma 1.19: Given free htpy $f_t : X \to Y$ and $x_0 \in X$. Let $h : I \to Y$, $t \mapsto f_t(x_0)$.

Then

$$\begin{tikzcd}
 & \pi_1(Y, f_1(x_0)) \arrow[dd, "\beta_h", "\cong"'] \\
\pi_1(X, x_0) \arrow[ur, "(f_1)_*"] \arrow[dr, "(f_0)_*"'] & \\
{[g]} & \pi_1(Y, f_0(x_0))
\end{tikzcd}$$

Commutes.



Proof: Let g be a loop in X at

Define $h_t(s) = h(ts)$.

Then $h_t \cdot f_t g \cdot \bar{h}_t$ is a htpy at $f_0 x_0$ from

$$h_0 \cdot f_0 g \cdot \bar{h}_0 \simeq f_0 g$$

to

$$h_1 \cdot f_1 g \cdot \bar{h}_1 = h \cdot f_1 g \cdot \bar{h}$$

So

$$(f_0)_*[g] = [h \cdot f_1 g \cdot \bar{h}]$$
$$= \beta_h((f_1)_*[g])$$

x_0.

of loops

. □

Pf of 1.18: Let g be a htpy inverse to f. Consider

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[rrd, "id_* = id"'] & \pi_1(Y, f(x_0)) \arrow[r, "g_*", "\cong"'] \arrow[d, phantom, "1.19 \checkmark"] & \pi_1(X, g(f(x_0))) \\
 & & \pi_1(X, x_0) \arrow[u, "\beta_h"', "\cong"]
\end{tikzcd}$$

where $h(t) = k_t(x_0)$, where k_t is the htpy from gf

Since $g_* f_*$ is iso, g_* is surjective.

Similarly, $f_* g_*$ is iso (different f_*), so g_* is

$\therefore$ g_* is an iso.

Since $g_* f_*$ is iso, $\underset{\uparrow \text{first}}{f_*}$ is iso. □

Exercises: 1.1.6, 1.1.16, 1.1.17.

$$\begin{tikzcd} & \pi_1(Y, f(x_0)) \arrow[d, "\cong"', "\beta_h"] \\ \arrow[ur, "id"] \arrow[r, "f_*"] & \pi_1(Y, f(g(f(x_0)))) \end{tikzcd}$$

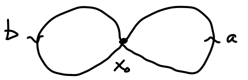
$$fg \underset{\ell}{\simeq} id$$

$\simeq id$.

, injective.

§ 1.2 Seifert - van Kampen Thm

What is $\pi_1(X, x_0)$ for

$X =$ 

$\subseteq \mathbb{R}^2$.

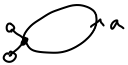
$a, b \in \pi_1(X, x_0)$.

So get $a^2 b^{-3} a b^2$.

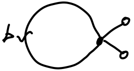
Thm: a and b generate $\pi_1(X, x_0)$.

Pf: Let f be a loop in X at x_0.

Let $A, B \subseteq X$ be open subsets:



A



B

Choose $0 = s_0 < s_1 < \cdots < s_m = 1$

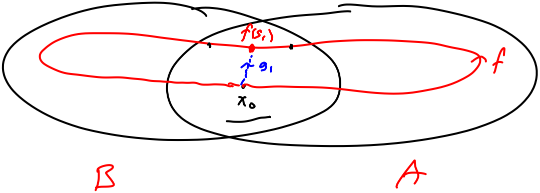
s.t. $f([s_{i-1}, s_i]) \subseteq A$ or B

(compactness). Call it A_i.

Write f_i for $f|_{[s_{i-1}, s_i]}$, scaled to $[0,1]$.

So $f \simeq f_1 \cdot f_2 \cdot \cdots \cdot f_m$.

For $i = 1, \ldots, m-1$, let g_i be a path in $A_i \cap A_{i+1}$ from x_0 to $f(s_i)$. (A, B, $A \cap B$ all path conn)



Then $f \simeq f_1 \cdot \ldots$

$\simeq (f_1 \cdot \bar{g}_1) \cdot$

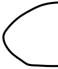
a composite of lo

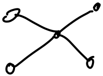
ith loop lands

So ith loop $\simeq$

for some $k \in \mathbb{Z}$.

to a word in





$A \cap B \simeq *$

$\cdot f_m$

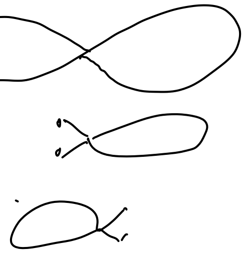
$(g_1 \cdot f_2 \bar{g}_1) \cdots (g_{m-1} \cdot f_m)$

ops at x_0.

in $A_i \simeq S^1$.

a^k or b^k

So f is $\simeq$

n and b. $\square$