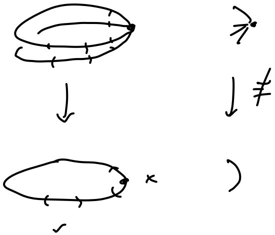
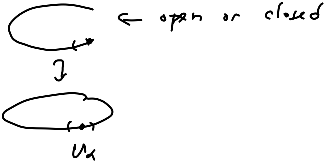
§1.3 Covering spaces (cont)

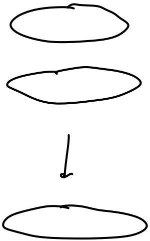
Ex: X, Y discrete. Any function $f: X \to Y$ is a cover.

Non-examples:

- 

- 

Ex:
$$\begin{tikzcd} \emptyset \arrow[d] \\ X \end{tikzcd}$$
✓

Ex: 

Ex: Any disjoint union of

$$\left\{ \begin{tikzcd} \tilde{X}_\alpha \arrow[d, "p_\alpha"'] \\ X \end{tikzcd} \right\} \rightsquigarrow$$

covers $\Rightarrow$

$$\begin{tikzcd} S^1 \sqcup S^1 \arrow[d, "\nabla"] \\ S^1 \end{tikzcd}$$

Covers of X.

$$\begin{tikzcd} \coprod_\alpha \tilde{X}_\alpha \arrow[d, "p"] \\ X \end{tikzcd}$$

p induced by p_α's.

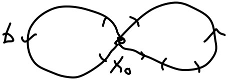
cover

Exercise 1.3.1: $\tilde{X} \xrightarrow{p} X$ cover, $A \subseteq X$

$\Rightarrow$ $\tilde{X}|_A := p^{-1}(A) \xrightarrow[p]{} A$ is a cover.

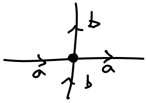
Covers of $S^1 \vee S^1$

$X = S^1 \vee S^1$



Defn: A **2-oriented graph**: a graph with:

- edges oriented (directed)
- edges labelled with "a" or "b"
- near each vertex, looks like



Any graph $\tilde{X}$ gives a space, also called $\tilde{X}$.

Extra data gives a cts map

$p : \tilde{X} \to X$:

- vertices $\mapsto$ x_0
- edges $\mapsto$ left or right loop depending on label.

Fact: This is a cover.

Get every cover of X in this way, up to iso. Iso of 2-oriented graphs iff covers iso.

Note: # of vertices = # a edges

= # b edges

Call this the *number of sheets*.

Exercise 1.3.10: Find

and 3-sheeted covers of

Choose $\tilde{x}_0 \in p^{-1}(x_0) =$

p induces

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow$$

$\|$

free group on $k+1$ gens, where k = # vertices

Fact: p_* is always i

all 2-sheeted

X up to iso.

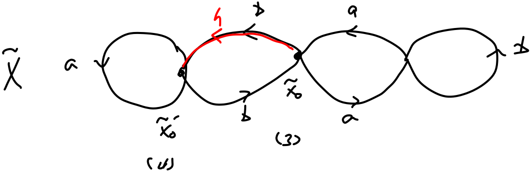
{vertices}.

$\pi_1(X, x_0)$

$\parallel$

$\langle a, b \rangle$

njective. $\|$



$h : \tilde x_0 \to \tilde x_0'$

$$\beta_h : \pi_1(\tilde X, \tilde x_0') \xrightarrow{\cong} \pi_1(\tilde X, \tilde x_0)$$

$$[f] \longmapsto [\underset{\sim}{b} \cdot f \cdot \underset{\sim}{b}^{-1}]$$

So

$$b\, p_*(\pi_1(\tilde X, \tilde x_0'))\, b^{-1} = p_*(\pi_1(\tilde X, \tilde x_0))$$

Mai

Fix

and

Then

a

n classification thm 1.38:

path conn, locally path conn,
semilocally simply conn (X, x_0).

$$\left\{\begin{array}{c}\text{connected covers}\\ \text{of } X\end{array}\right\}\Big/\text{iso} \cong \left\{\begin{array}{c}\text{subgps of}\\ \pi_1(X, x_0)\end{array}\right\}\Big/\text{conjugacy}$$

$$\tilde{X} \xrightarrow{p} X \longmapsto \text{im}(p_* \text{ for } \tilde{x}_0)$$

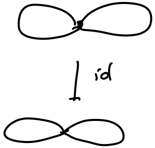
and

$$\left\{\begin{array}{c}\text{conn covers of } X\\ \text{with a choice of}\\ \tilde{x}_0\end{array}\right\}\Big/\begin{array}{c}\text{isos}\\ \text{preserving}\\ \text{basepoint}\end{array} \cong \left\{\begin{array}{c}\text{subgps of}\\ \pi_1(X, x_0)\end{array}\right\}$$

$$(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0) \longmapsto \text{im}(p_*)$$

re bijections

$X = S^1 \vee S^1$

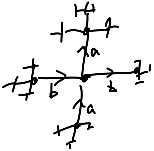


$\text{im}(p_*) =$

What cover corresponds to

$0 \subseteq \pi_1(X, x_0)$?

$\tilde{X}$ must be simply conn.



∞-graph with no loops.

"Universal cover".

Lifting

Fix a cover $p : \tilde{X} \to X$ (any X).

Def: A lift of $f : Y \to X$ is $\tilde{f} : Y \to \tilde{X}$ s.t.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes.

Prop 1.30: (Homotopy

htpy $\{f_t : Y \to X\}$

$\exists!$ htpy $\tilde{f}_t$

extended $\tilde{f}_0$.

Pf: The proof of (c

When $Y = pt$, get

$$\begin{tikzcd} & \emptyset \arrow[d] \\ I \arrow[ur, dashed, "\cancel{B}"] \arrow[r] & X \end{tikzcd}$$

Recall: If f_t i

then $\underline{\underline{\tilde{f}_t}}$ is a

$s = 0$

liftings) Given a

and lift $\tilde{f}_0$ of f_0,

lifting f_t and

) in Thm 1.7 applies. □

path lifting property.

s a htpy of paths,

htpy of paths.

Prop 1.31: Choose $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(\tilde{x})$.

(1) $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

(2) Let f_0 be a loop in X at x_0. Then $[f_0] \in \operatorname{im} p_*$ iff the unique lift $\tilde{f}_0$ starting at $\tilde{x}_0$ is a loop.

Pf: (1) Suppose $[\tilde{f}_0] \in \ker p_*$, so $\tilde{f}_0$ is a loop at $\tilde{x}_0$ and $f_0 := p\tilde{f}_0$ is null homotopic. So $\exists$ htpy f_t of loops with f_1 constant. By htpy lifting, $\exists$ htpy $\tilde{f}_t$ of loops. By uniqueness, f_1 is constant.

(2) $\Longleftarrow$: clear.

$\Longrightarrow$: $[f_0] \in \operatorname{im}(p_*)$

$\Longrightarrow$ $f_0 \simeq f_1$ via f_t, with

$f_1 = p\tilde{f}_1$ for $\tilde{f}_1$ a loop at $\tilde{x}_0$.

By htpy lifting, $\exists$ lifted htpy $\tilde{f}_t$ of loops. So $\tilde{f}_0$ is a loop. $\square$

Def: $|p^{-1}(x)|$ is a locally const. fn on X. So if X is connected, then it is constant. This is called the number of sheets of p.

Prop 1.32: Assume X, $\tilde{X}$ path conn, $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(X)$. Let $G := \pi_1(X, x_0)$ and $H := \operatorname{im}(\pi_1(\tilde{X}, \tilde{x}_0)) \leq G$. Then

# of sheets $= [G : H] = |H \backslash G|$.

Pf: Define

$$\Phi : H\backslash G \longrightarrow$$

$$H[g] \longmapsto$$

where g is a loop

and $\tilde{g}$ is the lift

Well-defined:

- $\tilde{g}(1)$ depends
  since htpy
  of paths.
- $\widetilde{h \cdot g} = \tilde{h} \cdot$
  loop at $\tilde{x}_0$,
  $([h] \in H)$.
  So $\widetilde{h \cdot g}$ (

$\longrightarrow p^{-1}(x_0)$

$\longrightarrow \tilde{g}(1)$

in X at x_0

starting at $\tilde{x}_0$.

only on $[g]$

lift to a htpy

$\tilde{g}$, since $\tilde{h}$ is a

by 1.31(2).

$)) = \tilde{g}(1)$.

Surjective: Given $\tilde{x} \in p^{-1}(x_0)$.

Choose path $\tilde{g}$ from $\tilde{x}_0$ to $\tilde{x}$.

Then $g := p\tilde{g}$ is a loop at x_0.

And $\Phi(H[g]) = \tilde{g}(1) = \tilde{x}_0$.

injective: $\Phi(H[g_1]) = \Phi(H[g_2])$

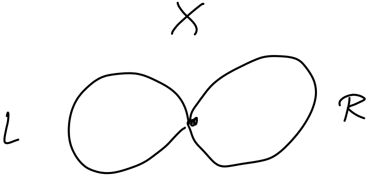
$\Longrightarrow \tilde{g}_1(1) = \tilde{g}_2(1)$

$\Longrightarrow g_1 \cdot \bar{g}_2$ lifts to a loop

$(\tilde{g}_1 \cdot \bar{\tilde{g}}_2)$

$\Longrightarrow [g_1][g_2]^{-1} \in H$

$\Longrightarrow H[g_1] = H[g_2]$. $\square$



Suppose we have a def. retraction H

from X onto L.

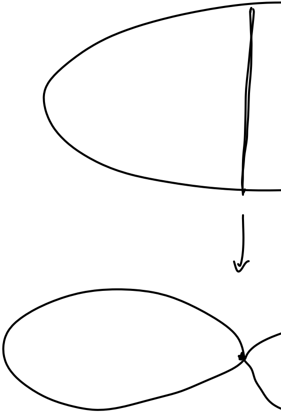
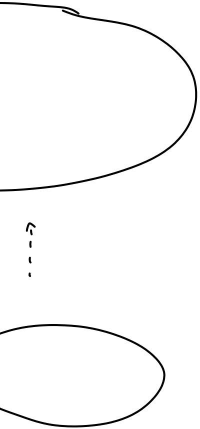
Write r for the retraction onto R

$r|_R = id_R$

$r|_L$ = constant at x_0.

Then $rH|_R$ is a def. retraction

from R to x_0.

$\pi_1 \mathbb{Z}$ π_1 triv.