Exercise 1.2.8

1.2.8: Compute the fundamental group of the space X obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

[Draw picture.]

Solution: Write T_1 and T_2 for the two tori and S_i for $S^1 \times \{x_0\}$ in T_i, $i = 1, 2$.

Idea: $X = T_1 \cup T_2$, so use SVK. But need an open cover.

For $i = 1, 2$, let U_i be a small, open neighbourhood of S_i in T_i that deformation retracts onto S_i, e.g. $U_i = S^1 \times V$, where V is an open interval around x_0 in S^1.

Let $A_1 := T_1 \cup U_2$ and $A_2 := T_2 \cup U_1$ in X, so $X = A_1 \cup A_2$.

Basepoint: (x_0, x_0) is in each A_i.

Note that $A_i \simeq T_i$, because U_{2-i} deformation retracts onto S_{2-i}.

A_1 and A_2 are open. For example, the preimage of A_1 in $T_1 \coprod T_2$ is $T_1 \coprod U_2$, which is open.

They are also path connected, since any point in T_i has a path to $(x_0, x_0) \in S_i$, as does any point in U_i, as U_i deformation retracts to S_i and $S_i \cong S^1$ is path connected.

Now, $A_1 \cap A_2$ is equal to $U_1 \cup U_2$, which deformation retracts onto the circle along which they are glued. So $A_1 \cap A_2 \simeq S^1$.

In particular, the pairwise (and triple) intersections are path connected.

So we can apply SVK to this cover, and obtain

$$\begin{aligned}
\pi_1(X) &\cong (\pi_1(A_1) * \pi_1(A_2))/N \qquad \text{all based at } (x_0, x_0) \\
&\cong (\pi_1(T_1) * \pi_1(T_2))/N \\
&\cong (\langle a, b \mid ab = ba \rangle * \langle c, d \mid cd = dc \rangle)/N \\
&\cong \langle a, b, c, d \mid ab = ba, cd = dc \rangle/N,
\end{aligned}$$

where N is the normal subgroup generated by equating the images of elements of $\pi_1(A_1 \cap A_2) \cong \pi_1(S^1) \cong \mathbb{Z}$.

It suffices to deal with the generator of $\pi_1(A_1 \cap A_2)$.

Choose the generators a, b, c, d so that, say, a and c correspond to the loop within S_1 and S_2, respectively. Then we get

$$\begin{aligned}
\pi_1(X) &\cong \langle a, b, c, d \mid ab = ba, cd = dc \rangle/N \\
&\cong \langle a, b, c, d \mid ab = ba, cd = dc, a = c \rangle \\
&\cong \langle a, b, d \mid ab = ba, ad = da \rangle.
\end{aligned}$$