Representation theory, Homework Set 1

Due Friday, May 10, at the beginning of class. These questions use material up to and including the lecture on Friday, May 3.

Remember that homework is graded on both **clarity** and correctness, and that clarity means giving an appropriate amount of detail, not too much and not too little. Good notation and a good choice of proof strategy help as well.

Please leave space in your solutions for my comments. Leave **large margins**, and leave at least 5cm between solutions. (Even better: start each solution on a new page.) Staple in the top left corner (unless electronic).

Also remember to write up your solutions on your own, and to not look at other students' solutions or search for solutions online or in other textbooks.

Q1 (2 marks): Let D_4 denote the dihedral group of order 8. Thought of as a subgroup of $O(2)$, it is generated by r and s, where r is rotation of $\mathbb{R}^2$ counterclockwise by $\pi/2$ and s is reflection in the x-axis. Let $\varphi : D_4 \to GL_2(\mathbb{C})$ be the representation given by

$$\varphi(r^k) = \begin{bmatrix} i^k & 0 \\ 0 & (-i)^k \end{bmatrix}, \qquad \varphi(sr^k) = \begin{bmatrix} 0 & (-i)^k \\ i^k & 0 \end{bmatrix}.$$

Prove that φ is irreducible. (You don't need to prove that the given formulas define a representation.)

Q2: (a) (2 marks) Let $\varphi_1, \varphi_2 : G \to \mathbb{C}^*$ be one-dimensional representations in "matrix" form. Show that φ_1 is equivalent to φ_2 if and only if $\varphi_1 = \varphi_2$.

(b) (1 mark) If $\varphi_i : G \to GL(V_i)$, for $i = 1, 2$, are equivalent representations of G, and V_1 and V_2 are one-dimensional, does it follow that $\varphi_1 = \varphi_2$?

Q3 (4 marks): Exercise 3.4.

Q4 (3 marks): Exercise 4.10. (No character theory is needed for this. You can work directly from the definition.)

Q5(a) (3 marks) Exercise 3.8.1.

(b) (2 marks) Instead of 3.8.2, show that if V and W are irreducible G-representations with dimensions greater than 1, and $\varphi : G \to GL(V \oplus W)$ is their direct sum, then the maps φ_g for $g \in G$ do not have a common eigenvector. (But $V \oplus W$ is clearly reducible.)