Representation theory, Homework Set 2

Here is the complete list of questions for HW2, which is due May 22 at 10am, using **GradeScope**.

Please put each solution on its own page.

Remember that homework is graded on both **clarity** and correctness, and that clarity means giving an appropriate amount of detail, not too much and not too little. Good notation and a good choice of proof strategy help as well.

Q1 (4 marks): Let p be a prime and let k be the field with p elements. Based on Example 3.2.6, find a representation of $\mathbb{Z}/p$ on a k-vector space which is not completely reducible. Explain why it is a representation, why it is not completely reducible, and explain where the proof of Maschke's Theorem breaks down.

Q2: Consider the representation of $G = \mathbb{Z}/4$ on $V = \mathbb{R}^2$, where the generator acts via rotation by 90 degrees counterclockwise. This is a *real* representation. (You don't need to prove this.)

(a) (1 mark) Show that it is irreducible.
(b) (2 marks) Show that $\dim_{\mathbb{R}} \operatorname{Hom}_G(V, V) > 1$.

Q3: Consider the equilateral triangle in $\mathbb{R}^2$ with vertices

$$v_1 = (1, 0),\ v_2 = (-1/2, \sqrt{3}/2) \quad \text{and} \quad v_3 = (-1/2, -\sqrt{3}/2).$$

The group S_3 permutes these vertices by $\sigma(v_i) = v_{\sigma(i)}$ and this extends linearly to give an action φ on $\mathbb{R}^2$. Consider the matrix form of this action with respect to the standard basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and consider these matrices as being in $GL_2(\mathbb{C})$, so $\varphi\colon S_3 \to GL_2(\mathbb{C})$. Since the action preserves lengths, the matrices are unitary. For example, since $\varphi((23))$ is reflection in the x-axis,

$$\varphi((23)) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(a) (2 marks) Compute all six matrices of φ.
(b) (2 marks) Verify for yourself that the four functions $\varphi_{ij} : G \to \mathbb{C}$ you obtain are orthogonal and have the correct norm. Only show me the computations for $\langle \varphi_{11}, \varphi_{11} \rangle$ and $\langle \varphi_{11}, \varphi_{12} \rangle$.
(c) (2 marks) Use the character χ_φ to show that φ is irreducible and is isomorphic to the 2-dimensional representation ρ from Example 3.1.14.

Q4: Let $\alpha\colon S_n \to GL_n(\mathbb{C})$ be the standard representation from Example 3.1.9.

(a) (1 mark) Show that $\chi_\alpha(\sigma)$ equals the number of i such that $\sigma(i) = i$.
(b) (2 marks) (The remaining parts take $n = 4$.) List one representative of each conjugacy class of S_4 and compute the number of elements in each conjugacy class.
(c) (2 marks) Compute χ_α on each representative from (b) and determine how many copies of the trivial representation occur in α.
(d) (2 marks) Use $\langle \chi_\alpha, \chi_\alpha \rangle$ to determine how many other irreps occur in α, and deduce what their characters must be.