Representation theory, Homework Set 3

Due Friday, May 31 at 10am, using Gradescope.

We have already covered all of the relevant material.

Q1 (1+2+1+2 marks): Exercise 4.11. (If needed for your solution, be sure to show that the representations described in the question are not isomorphic. Note that $M_{mn}(\mathbb{C})$ means $m \times n$ matrices.)

Q2 (2+2+1+1 marks): Exercise 4.12. (For part 1, be sure to show that ρ is a representation. For part 3, you can save work, since you know there are five inequivalent irreps. Note that the quaterion group is the group generated by symbols $\hat{\imath}, \hat{\jmath}, \hat{k}$ with the relations $\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = \hat{\imath}\hat{\jmath}\hat{k}$. The symbol -1 is then *defined* to be the common value, and it *follows* that (-1) is central and $(-1)^2 = 1$.)

Q3 (0 marks): Exercise 5.6. (Not graded, but a good exercise to do.)

Q4 (4 marks): Exercise 5.13.1. (Part 1 only. Show it is well-defined and an isomorphism of groups.)

Q5 (2+2 marks): Consider the action of $G \times G$ on G by $(g,h)k = gkh^{-1}$. This gives a representation of $G \times G$ on the group algebra $\mathbb{C}G$. Since $\mathbb{C}G$ is isomorphic to $L(G)$, $L(G)$ also has an action of $G \times G$ on it.

(a) Describe explicitly how $G \times G$ acts on $L(G)$.

Now suppose that $\varphi : G \to GL_d(\mathbb{C})$ is a representation of G. Then $M_d(\mathbb{C})$ is a representation of $G \times G$ via

$$(g,h)A = \varphi_g A \varphi_h^{-1}$$

[This is similar to 4.11, but slightly different.]

(b) Show that the map $T : L(G) \to M_d(\mathbb{C})$ sending f to

$$\sum_{g \in G} f(g)\, \varphi_g$$

is a $G \times G$ morphism.

The rest is **not** homework, but is for your entertainment.

If φ is an irreducible G rep, then the above makes $M_d(\mathbb{C})$ into an irreducible $G \times G$ rep (proof similar to 4.11), and so one way to express Wedderburn's Thm (5.5.6) is that we have decomposed $L(G)$ into irreps with respect to the $G \times G$ action.

Moreover, if you only look at how the left factor of $G \times G$ acts, you see that $M_d(\mathbb{C})$ is isomorphic as a G-rep to d copies of the representation φ. Combining this with the isomorphism of $\mathbb{C}G$ with $L(G)$, one recovers the decomposition of the regular representation into d_k copies of each irrep $\varphi^{(k)}$. So Wedderburn's Theorem is a stronger version of this decomposition, which also takes into account the right action of G on itself.