Linear Algebra 040a Final Examination Tuesday, December 13, 2005

1. [*4 marks*] Find an equation of the line in $\mathbf{R}^4$ that is parallel to $(3, -1, 1, 2)$ and passes through $(5, 1, 0, -2)$.
2. [*6 marks*] Find the area of the triangle with vertices $(2, -1, 4)$, $(5, -3, 5)$, and $(-2, 2, 2)$.
3. [*4 marks*] Write down the elementary matrix E that satisfies $EA = B$ where

$$A = \begin{bmatrix} 5 & 1 & 9 \\ 2 & 4 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 0 \\ 5 & 1 & 9 \\ -1 & 3 & -2 \end{bmatrix}.$$

4. [*6 marks*] Can the vector $(3, 1, 1)$ be expressed as a linear combination of the vectors $(2, 5, -1)$, $(1, 6, 0)$, $(5, 2, -4)$? Justify your answer.
5. [*8 marks*] Evaluate the determinant of

$$\begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \\ 1 & 3 & 1 & 3 & 1 \\ 3 & 1 & 3 & 1 & 3 \end{bmatrix}.$$

6. [*10 marks*] Find an orthonormal basis for the span of the vectors $(2, 1, -2)$, $(5, 7, 4)$, $(1, 0, -2)$.
7. [*6 marks*] What is the rank of

$$\begin{bmatrix} 3 & 1 & 0 & -1 & 5 & 0 & 4 \\ 2 & 0 & 1 & 5 & -1 & 0 & 2 \\ -7 & 0 & 0 & 2 & 4 & 1 & 3 \end{bmatrix}?$$

   Justify your answer.
8. [*10 marks*] Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 - x_2 + x_3).$$

   If $\mathcal{B} = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ and $\mathcal{B}' = \{(2, 1), (1, -2)\}$ find $[T]_{\mathcal{B}', \mathcal{B}}$.
9. [*10 marks*] Let

$$A = \begin{bmatrix} -1 & -2 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

   Find a diagonal matrix D and an invertible matrix P such that $AP = PD$.
10. Let the operator $S : \mathbf{R}^3 \to \mathbf{R}^4$ be defined by

$$S(x_1, x_2, x_3) = (x_1 - 4x_2 + 2x_3, 2x_1 + 7x_2 - x_3, -x_1 - 8x_2 + 2x_3, 2x_1 + x_2 + x_3).$$

   (a) [*2 marks*] Find the standard matrix of S.
   (b) [*6 marks*] Find a basis for the range of S.

11. [*8 marks*] Let

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 0 \\ 0 \end{bmatrix}.$$

Find the vector $\mathbf{x} \in \mathbf{R}^3$ for which $\|\mathbf{b} - A\mathbf{x}\|$ is a small as possible.

12. Circle the correct answers.

(a) [*2 marks*] If $\lambda = 0$ is an eigenvalue of A then $\lambda = 0$ is an eigenvalue of A^T. **T** **F**

(b) [*2 marks*] Every orthogonal operator on $\mathbf{R}^2$ is either a rotation or a reflection. **T** **F**

(c) [*2 marks*] If U and V are subspaces of $\mathbf{R}^n$ and $U \cap V = \{\mathbf{0}\}$ then $U = V^\perp$. **T** **F**

(d) [*2 marks*] A 5×9 matrix of rank 3 has a 6-dimensional null space. **T** **F**

(e) [*2 marks*] If A is a diagonalizable matrix then A^2 is also diagonalizable. **T** **F**

(f) [*2 marks*] If $\mathcal{B}$ is any basis of $\mathbf{R}^n$ and $\mathcal{B}'$ is an orthonormal basis of $\mathbf{R}^n$ then for all $\mathbf{x} \in \mathbf{R}^n$ the vectors $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}'}$ have the same length. **T** **F**

(g) [*2 marks*] It is possible to extend the set

$$\{(1,1,1,1,1),(1,2,3,4,5),(1,4,9,16,25)\}$$

to a basis of $\mathbf{R}^5$. **T** **F**

(h) [*2 marks*] If $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors in $\mathbf{R}^n$, and

$$\mathbf{w} = \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u}$$

then $\mathbf{w}$ is orthogonal to $\mathbf{u}$. **T** **F**

(i) [*2 marks*] If the columns of A are a basis of $\mathbf{R}^n$ then A must be an $n \times n$ matrix. **T** **F**

(j) [*2 marks*] The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ represents the orthogonal projection onto the x-axis. **T** **F**