Linear Algebra 040a Midterm Examination Friday, November 4, 2005

1. [*2 marks*] Find the dot product of the vectors $(1, 0, 0, -3, 2)$ and $(2, 3, 0, -2, -2)$.
2. [*2 marks*] Let θ be the angle between the vectors $(0, 1, 0)$ and $(2, 1, 2)$. Find $\cos\theta$.
3. [*2 marks*] Find the value of $c \in \mathbf{R}$ that makes the vectors $(1, 1, 2, 2)$ and $(-1, c, 3, -5)$ orthogonal.
4. [*2 marks*] Find a non-zero vector orthogonal to both $(2, 1, -1)$ and $(1, 2, 1)$.
5. [*4 marks*] Solve the following system of equations.

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= 1 \\ x_1 + x_2 - x_3 &= 0 \\ 3x_1 + 2x_2 - 3x_3 &= 1 \end{aligned}$$

6. [*6 marks*] Let

$$B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 5 & 2 \end{bmatrix} \quad \text{and } D = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & 2 \end{bmatrix}.$$

Exactly two of the following four expressions are defined and the rest are not. Find the value of each defined expression.

$$BC, \quad B(C^T + D), \quad D^T - C^T, \quad CD.$$

7. [*6 marks*] Solve the linear system $A\mathbf{x} = \mathbf{0}$ where

$$A = \begin{bmatrix} 2 & 3 & -1 & 4 \\ 1 & -1 & 2 & 2 \\ 1 & 2 & -1 & 2 \end{bmatrix}.$$

Express your answer in vector form.

8. [*6 marks*] Find the inverse of the matrix

$$\begin{bmatrix} 2 & 3 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -4 & 2 \end{bmatrix}.$$

9. [*6 marks*] Let

$$M = \begin{bmatrix} 3 & -1 & 1 & 5 \\ 6 & 0 & 4 & 0 \\ 2 & 1 & 1 & 2 \\ 0 & -1 & 5 & 1 \end{bmatrix}.$$

Find C_{24}, the $(2, 4)$-cofactor of M.

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) [*2 marks*] Express $\mathrm{tr}(A)$, the trace of A, in terms of a, b, c, and d.

(b) [*2 marks*] Express the matrix A^2, in terms of a, b, c, and d.
(c) [*2 marks*] Express $\mathrm{tr}(A^2)$, in terms of a, b, c, and d.
(d) [*2 marks*] Evaluate and simplify

$$\frac{1}{2}\det\begin{bmatrix} \mathrm{tr}(A) & 1 \\ \mathrm{tr}(A^2) & \mathrm{tr}(A) \end{bmatrix}.$$

11. [*6 marks*] Let $\mathbf{x} = (x, y, z)$, $P = (1, 1, 2)$, $Q = (3, -2, 4)$ and $R = (2, 2, -3)$. Work out both sides of the equation

$$\det\begin{bmatrix} \mathbf{x} \\ Q-P \\ R-P \end{bmatrix} = \det\begin{bmatrix} P \\ Q-P \\ R-P \end{bmatrix}$$

and verify that the resulting equation represents a plane passing through P, Q, and R.

12. [*12 marks*] Find all eigenvalues of the matrix

$$\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

and find an eigenvector for each eigenvalue.

13. Let

$$A = \frac{1}{25}\begin{bmatrix} 20 & 0 & -15 \\ -9 & 20 & -12 \\ 12 & 15 & 16 \end{bmatrix}.$$

(a) [*2 marks*] Write down A^T.
(b) [*4 marks*] Calculate $A^T A$.
(c) [*2 marks*] Is the matrix A an orthogonal matrix? Justify your answer.

14. Suppose that A is an invertible 5×5 matrix and some but not all of the entries of A^{-1} are known. Specifically,

$$A^{-1} = \begin{bmatrix} 2 & * & -4 & * & 1 \\ 0 & -1 & * & 2 & -5 \\ 0 & 0 & 3 & 4 & * \\ 0 & 0 & 0 & 2 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) [*2 marks*] Find the determinant of A.
(b) [*2 marks*] Find C_{43}, the $(4,3)$-cofactor of A. (Hint: consider the adjoint of A.)

15. Suppose $T : \mathbf{R}^3 \to \mathbf{R}^4$ is a linear transformation that satisfies

$$\begin{aligned} T(\mathbf{e}_1) &= (1, -1, 0, 0), \\ T(\mathbf{e}_1 + \mathbf{e}_2) &= (2, 2, -4, 0), \\ T(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) &= (3, 3, 3, -9). \end{aligned}$$

Here $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.
(a) [*4 marks*] Find the standard matrix of T.

(b) [*4 marks*] Let $\mathbf{a} = (1,1,1,1) \in \mathbf{R}^4$. Show that $\operatorname{ran}(T) = \mathbf{a}^\perp$, that is, show that the range of T is equal to $\{\mathbf{x} \in \mathbf{R}^4 : \mathbf{a} \cdot \mathbf{x} = 0\}$.

16. Circle the correct answers.

(a) [*2 marks*] If $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ and W is a subspace of $\mathbf{R}^n$ containing $\mathbf{u} - \mathbf{v}$ then both $\mathbf{u}$ and $\mathbf{v}$ are in W. **T** **F**

(b) [*2 marks*] The only 8×8 matrix that is symmetric and anti-symmetric is the 8×8 zero matrix. **T** **F**

(c) [*2 marks*] If $\mathbf{u} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ then $-\mathbf{u} \in \operatorname{span}\{-\mathbf{v}_1, -\mathbf{v}_2, \ldots, -\mathbf{v}_n\}$. **T** **F**

(d) [*2 marks*] If A and B are 9×9 matrices then any solution of $A\mathbf{x} = \mathbf{0}$ is also a solution of $(AB)\mathbf{x} = \mathbf{0}$. **T** **F**

(e) [*2 marks*] The vectors $(3,0,1,0)$, $(-1,0,2,-1)$, and $(3,0,4,1)$ are linearly dependent. **T** **F**

(f) [*2 marks*] If T is a linear operator on $\mathbf{R}^7$ then $T(x) \cdot T(y) = x \cdot y$ for all $x, y \in \mathbf{R}^7$. **T** **F**

(g) [*2 marks*] If the column space of the 6×13 matrix A is equal to $\mathbf{R}^6$ then the $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbf{R}^6$. **T** **F**

(h) [*2 marks*] If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set of vectors in $\mathbf{R}^9$ then

$$\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 + 4\mathbf{v}_4 \neq 4\mathbf{v}_1 + 3\mathbf{v}_2 + 3\mathbf{v}_3 + \mathbf{v}_4.$$

T **F**

(i) [*2 marks*] Every elementary matrix has non-zero determinant. **T** **F**