Section 1.1: Vectors and Matrices; n-space

Definition 1.1.2. An **ordered *n*-tuple** is a sequence of n real numbers $(v_1, v_2, \dots, v_n)$. This is also called a **vector**. $\mathbb{R}^n$ denotes the set of all ordered n-tuples and is called ***n*-space**.

Theorem 1.1.5. *For* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and* $k, l \in \mathbb{R}$:

(*a*) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(*b*) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(*c*) $\mathbf{u} + \mathbf{0} = \mathbf{u} = \mathbf{0} + \mathbf{u}$

(*d*) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(*e*) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$

(*f*) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(*g*) $k(l\mathbf{u}) = (kl)\mathbf{u}$

(*h*) $1\mathbf{u} = \mathbf{u}$

Section 1.2: Dot product and orthogonality

Definition 1.2.1. If $\mathbf{v} = (v_1, v_2, \dots, v_n)$ is a vector in $\mathbb{R}^n$, then the **length** of $\mathbf{v}$, also called the **norm** of $\mathbf{v}$ or the **magnitude** of $\mathbf{v}$, is

$$\|v\| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}. \tag{3}$$

Definition 1.2.5. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, the **dot product** is the scalar

$$\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n. \tag{12}$$

Note.

$$\mathbf{v} \cdot \mathbf{v} = v_1 v_1 + \cdots + v_n v_n = \|\mathbf{v}\|^2, \tag{13}$$

so

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}. \tag{14}$$

Theorem 1.2.8. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are nonzero vectors in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*, then*

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\| \cos\theta, \tag{15'}$$

where θ *is the angle between them* $(0 \le \theta \le \pi)$.

Theorem 1.2.12 (Cauchy-Schwarz inequality). *For* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|. \tag{22}$$

Definition. The **angle** θ between nonzero vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is *defined* to be

$$\theta := \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right). \tag{19}$$

In particular, $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthogonal** (or **perpendicular**) if $\mathbf{u} \cdot \mathbf{v} = 0$ (so $\theta = \pi/2 = 90°$).

Definition 1.2.10. Vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthonormal** is they are orthogonal and have length 1. A set of vectors is **orthonormal** is each vector in the set has length 1 and each pair of vectors is orthogonal.

Basic Properties of dot product.

Recall: $\mathbf{u} \cdot \mathbf{v} := u_1 v_1 + \cdots + u_n v_n$.

Theorem 1.2.6. *For* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *and* $k \in \mathbb{R}$:

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ *(and the reverse)*

(*c*) $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

(*d*) $\mathbf{v} \cdot \mathbf{v} \geq 0$

(*e*) $\mathbf{v} \cdot \mathbf{v} = 0$ *if and only if* $\mathbf{v} = \mathbf{0}$.

Theorem 1.2.7.

(*a*) $\mathbf{0} \cdot \mathbf{v} = 0 = \mathbf{v} \cdot \mathbf{0}$

(c) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$ *(and the reverse)*

Theorem 1.2.11 (Pythagorean theorem). *If* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ *are orthogonal, then*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2. \tag{18}$$

Theorem 1.2.13 (triangle inequality). *For any* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|. \tag{26}$$

Section 1.3: Vector equations of lines and planes

The **general equation** of a line in $\mathbb{R}^2$ is

$$Ax + By = C \qquad (A \text{ and } B \text{ not both zero}). \tag{1}$$

For $\mathbf{x}_0, \mathbf{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$, the **vector equation** of the line through $\mathbf{x}_0$ and parallel to $\mathbf{v}$ is

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}, \tag{4}$$

where $-\infty < t < \infty$ is the **parameter**.
If $\mathbf{x} = (x, y, z)$, $\mathbf{x}_0 = (x_0, y_0, z_0)$ and $\mathbf{v} = (a, b, c)$, the **parametric equations** are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct. \tag{7}$$

(Similar in $\mathbb{R}^2$.)
The **point-normal equation** of the plane in $\mathbb{R}^3$ normal to $\mathbf{n} = (A, B, C)$ and through $\mathbf{x}_0 = (x_0, y_0, z_0)$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{16}$$

which when expanded becomes the **general equation**

$$Ax + By + Cz = D. \tag{17}$$

The **vector equation** of the plane in $\mathbb{R}^3$ parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ and through $\mathbf{x}_0 = (x_0, y_0, z_0)$ is

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2, \tag{20}$$

where $-\infty < t_1 < \infty$, $-\infty < t_2 < \infty$ are the **parameters**.
If $\mathbf{x} = (x, y, z)$, $\mathbf{x}_0 = (x_0, y_0, z_0)$, $\mathbf{v}_1 = (a_1, b_1, c_1)$, $\mathbf{v}_2 = (a_2, b_2, c_2)$, the **parametric equations** of this plane are

$$\begin{aligned} x &= x_0 + a_1 t_1 + a_2 t_2 \\ y &= y_0 + b_1 t_1 + b_2 t_2 \\ z &= z_0 + c_1 t_1 + c_2 t_2 \end{aligned} \tag{22}$$