Section 3.4: Subspaces and Linear Independence

Definition 3.4.1. Let W be any set of vectors in $\mathbb{R}^n$.

(a) W is **closed under scalar multiplication** if $\mathbf{x} \in W \implies k\mathbf{x} \in W$ for every scalar $k \in \mathbb{R}$.

(b) W is **closed under addition** if $\mathbf{x}, \mathbf{y} \in W \implies \mathbf{x} + \mathbf{y} \in W$.

(c) W is a **subspace of $\mathbb{R}^n$** if both hold and W is non-empty.

For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s \in \mathbb{R}^n$, the **span** is the set of all linear combinations $t_1\mathbf{v}_1 + \cdots + t_s\mathbf{v}_s$ for all scalars $t_1, t_2, \ldots, t_s \in \mathbb{R}$. It is denoted $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$.

Theorem 3.4.2. $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ *is a subspace of* $\mathbb{R}^n$.

Definition 3.4.5. A non-empty set of vectors $S = \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s \in \mathbb{R}^n$ is **linearly independent** if

$$c_1\mathbf{v}_1 + \cdots + c_s\mathbf{v}_s = \mathbf{0}$$

only when $c_1 = c_2 = \cdots = c_s = 0$. Otherwise, S is **linearly dependent**.

Theorem 3.4.9 (Unifying Theorem). *If A is an $n \times n$ matrix, then the following statements are equivalent.*

(*a*) *The reduced row echelon form of A is I_n.*

(*b*) *A is expressible as a product of elementary matrices.*

(*c*) *A is invertible.*

(*d*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in* $\mathbb{R}^n$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in* $\mathbb{R}^n$.

(*g*) *The column vectors of A are linearly independent.*

(*h*) *The row vectors of A are linearly independent.*

Section 3.5: The Geometry of Linear Systems

Theorem 3.5.2. *A* ***general*** *solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding a* ***particular*** *solution of $A\mathbf{x} = \mathbf{b}$ to a* ***general*** *solution of $A\mathbf{x} = \mathbf{0}$.*

If A is a matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, then the **column space** of A is

$$\operatorname{col}(A) := \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}.$$

Theorem 3.5.5. *The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in \operatorname{col}(A)$.*

Section 3.6: Matrices with Special Forms

Theorem 3.6.1. (*a*) $(\text{upper } \Delta)^T = \text{lower } \Delta$ *and* $(\text{lower } \Delta)^T = \text{upper } \Delta$.

(*b*) $(\text{upper } \Delta)(\text{upper } \Delta) = \text{upper } \Delta$ *and* $(\text{lower } \Delta)(\text{lower } \Delta) = \text{lower } \Delta$.

(*c*) *A triangular matrix is invertible if and only if all diagonal entries are non-zero.*

(*d*) $(\text{upper } \Delta)^{-1} = \text{upper } \Delta$ *and* $(\text{lower } \Delta)^{-1} = \text{lower } \Delta$.

A matrix A is **symmetric** if $A^T = A$ and **skew-symmetric** if $A^T = -A$.

Theorem 3.6.2 and 3.6.4. *If A and B are symmetric matrices of the same size, then A^T, $A + B$ and kA are symmetric. If A is invertible, then A^{-1} is symmetric too.*

But AB is *not* symmetric in general.