A and B denote $n \times n$ matrices.

Section 4.1: Determinants; Cofactor Expansion

Definition 4.1.1. The **determinant** of A is defined to be the sum of all signed elementary products from A:

$$\det A = \sum \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n},$$

where the sum is over all permutations of the column indices. The sign of an elementary product is $+1$ if an even number of interchanges is needed to put the indices into their natural order, and -1 otherwise.

Theorem 4.1.2. *If A has a row or a column of zeroes, then* $\det A = 0$.

Theorem 4.1.3. *If A is a triangular matrix, then* $\det A$ *is the product of the entries on the main diagonal.*

Definition 4.1.4. The (i,j)-th **minor** M_{ij} of A is the determinant of the submatrix of A obtained by deleting the ith row and the jth column. The (i,j)-th **cofactor** of A is $C_{ij} = (-1)^{i+j} M_{ij}$.

Theorem 4.1.5. *Let* $1 \leq i \leq n$, $1 \leq j \leq n$.

Cofactor expansion along the jth column: $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

Cofactor expansion along the ith row: $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

Section 4.2: Properties of Determinants

Theorem 4.2.1. $\det A = \det A^T$.

Theorem 4.2.2. (*a*) *If B is the matrix obtained from A by multiplying a single row (column) by the scalar k, then* $\det B = k \det A$.

(*b*) *If B is the matrix obtained from A by interchanging two rows (columns), then* $\det B = -\det A$.

(*c*) *If B is the matrix obtained from A by adding a multiple of one row (column) to another row (column), then* $\det B = \det A$.

Theorem 4.2.3.(*a,b*) *If A has two identical or proportional rows (columns), then* $\det A = 0$.

(*c*) $\det(kA) = k^n \det A$.

Theorem 4.2.4. *A is invertible if and only if* $\det A \neq 0$.

Theorem 4.2.5. $\det(AB) = \det A \cdot \det B$.

Theorem 4.2.6. *If A is invertible, then* $\det(A^{-1}) = \dfrac{1}{\det A}$.

Theorem 4.2.7 (Unifying Theorem). *The following statements are equivalent.*

(*a*) *The reduced row echelon form of A is I_n.*

(*b*) *A is expressible as a product of elementary matrices.*

(*c*) *A is invertible.*

(*d*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every vector* $\mathbf{b}$ *in* $\mathbb{R}^n$.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in* $\mathbb{R}^n$.

(*g*) *The column vectors of A are linearly independent.*

(*h*) *The row vectors of A are linearly independent.*

(*i*) $\det A \neq 0$.

Section 4.3: Cramer's Rule; Formula for A^{-1}; Applications of Determinants

Theorem 4.3.5. *(a) If A is a 2×2 matrix, then $|\det A|$ represents the area of the parallelogram determined by the two column vectors (positioned so that their initial points coincide).*

(b) If A is a 3×3 matrix, then $|\det A|$ represents the volume of the parallelepiped determined by the three column vectors (positioned so that their initial points coincide).

Definition 4.3.7. The **cross product** of $\mathbf{u} = (u_1, u_2, u_3)$ with $\mathbf{v} = (v_1, v_2, v_3)$ is

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_2) = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right).$$

Theorem 4.3.8. *If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$ and k is a scalar, then:*

(a) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$.

(d) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$.

(e) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$.

(f) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

Theorem 4.3.9. *If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$:*

(a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

(b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

Theorem 4.3.10. *Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$, and let θ be the angle between them.*

(a) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$.

(b) The area of the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides is $A = \|\mathbf{u} \times \mathbf{v}\|$.

Section 4.4: A First Look at Eigenvalues and Eigenvectors

Definition 4.4.3. A scalar λ is called an **eigenvalue** of A if there is a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. If λ is an eigenvalue of A, then every non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvector** of A corresponding to λ.

Theorem 4.4.4. *Let λ be a scalar. The following statements are equivalent.*

(a) λ is an eigenvalue of A.

(b) λ is a solution of the equation $\det(\lambda I - A) = 0$.

(c) The linear system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has non-trivial solutions.

Theorem 4.4.5. *If A is a triangular matrix (upper or lower triangular or diagonal), then the eigenvalues of A are the entries of the main diagonal of A.*

Theorem 4.4.7 (Unifying Theorem). *(j) $\lambda = 0$ is not an eigenvalue of A.*

Theorem 4.4.8. *The characteristic polynomial of A can be expressed as*

$$\det(\lambda I - A) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the distinct eigenvalues of A, and the corresponding algebraic multiplicities satisfy $m_1 + m_2 + \cdots + m_k = n$.

Theorem 4.4.12. *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A (repeated according to multiplicity), then:*

(a) $\det A = \lambda_1\lambda_2 \cdots \lambda_n$.

(b) $\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.