Section 6.1: Matrices as Transformations

Definition 6.1.1. A function $f: D \to C$ is a rule that associates to each x in the **domain** D a unique **value** $y = f(x)$ in the **codomain** C. One says that f **maps** x into $f(x)$ and that $f(x)$ is the **image** of x under f. The **range** $\operatorname{ran}(f)$ of f is the set of all values $f(x)$.

Definition 6.1.2. A function $T: \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$ if for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars c one has:

(a) $T(c\mathbf{u}) = cT(\mathbf{u})$ (**homogeneity**)

(b) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (**additivity**)

If $n = m$, T is also called a **linear operator**.

Theorem 6.1.3. *If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then*

(a) $T(\mathbf{0}) = \mathbf{0}$

(b) $T(-\mathbf{u}) = -T(\mathbf{u})$

(c) $T(\mathbf{u}-\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$

Theorem 6.1.4. *Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear. If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard unit vectors in $\mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$, then $T(\mathbf{x}) = A\mathbf{x}$ where $A = [T(\mathbf{e}_1)\, T(\mathbf{e}_2)\, \cdots\, T(\mathbf{e}_n)]$ is the **standard matrix** for A.*

Section 6.2: Geometry of linear operators

Theorem 6.2.1. *If $T: \mathbb{R}^n \to \mathbb{R}^n$ is linear, then the following statements are equivalent:*

(a) $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ *for all* $\mathbf{x} \in \mathbb{R}^n$

(b) $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

Definition 6.2.2. A square matrix A is **orthogonal** if it is invertible and $A^{-1} = A^T$.

Theorem 6.2.3. *(a)–(c) Transposes, inverses and products of orthogonal matrices are orthogonal.*

(d) If A is orthogonal, then $\det A = \pm 1$.

Theorem 6.2.4 and Theorem 6.2.5. *Let A be an $n \times n$ matrix. The following statements are equivalent:*

(a) $A^T A = I$, *i.e., A is orthogonal*

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

(d) The column vectors of A are orthonormal.

(e) The row vectors of A are orthonormal.

Theorem 6.2.6. *A linear operator $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if its standard matrix is orthogonal.*

Theorem 6.2.7. *Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal linear operator. Then its standard matrix is expressible in the form*

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad H_{\theta/2} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

That is, T is either a rotation about the origin through the angle θ or a reflection about a line through the origin which makes an angle $\theta/2$ with the positive x-axis.

Section 6.3: Kernel and range

Definition 6.3.1. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear. The set of vectors mapped to $\mathbf{0}$ under T is called the **kernel** $\ker T$ of T.

Theorem 6.3.2. *The kernel of a linear transformation* $T\colon \mathbb{R}^n \to \mathbb{R}^m$ *is a subspace of* $\mathbb{R}^n$.

Theorem 6.3.3 and Definition 6.3.4. *The solution space of the homogeneous linear system* $A\mathbf{x} = \mathbf{0}$ *is called the* ***null space*** $\operatorname{null}(A)$ *of the matrix* A. *It is the kernel of the linear transformation corresponding to* A.

Theorem 6.3.7. *The range* $\operatorname{ran}(T)$ *of a linear transformation* $T\colon \mathbb{R}^n \to \mathbb{R}^m$ *is a subspace of* $\mathbb{R}^m$.

Theorem 6.3.8. *Let* A *be a matrix. The range of the linear transformation corresponding to* A *is the column space of* A.

Definition 6.3.9. A linear transformation is **onto** if its range equals the entire codomain.

Definition 6.3.10. A linear transformation is **one-to-one** $(1-1)$ if it maps distinct vectors of the domain into distinct vectors of the codomain.

Theorem 6.3.11. *Let* T *be a linear transformation. Then* T *is one-to-one if and only if* $\ker T = \{0\}$.

Theorem 6.3.12. *Let* A *be a matrix. The corresponding linear transformation is one-to-one if and only if* $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

Theorem 6.3.13. *Let* A *be an* $m \times n$ *matrix. The corresponding linear transformation is onto if and only if* $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $\mathbf{b} \in \mathbb{R}^m$.

Theorem 6.3.14 and 6.3.15 (Unifying Theorem).

- (a) *The reduced row echelon form of* A *is* I_n.
- (b) *A is expressible as a product of elementary matrices.*
- (c) *A is invertible.*
- ***(d)*** $\boldsymbol{A\mathbf{x} = \mathbf{0}}$ ***has only the trivial solution.***
- ***(e)*** $\boldsymbol{A\mathbf{x} = \mathbf{b}}$ ***is consistent for every vector*** $\boldsymbol{\mathbf{b}}$ ***in*** $\boldsymbol{\mathbb{R}^n}$.
- (f) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every vector* $\mathbf{b}$ *in* $\mathbb{R}^n$.
- (g) *The column vectors of A are linearly independent.*
- (h) *The row vectors of A are linearly independent.*
- (i) $\det A \neq 0$.
- (j) $\lambda = 0$ *is not an eigenvalue of* A.
- ***(k)*** $\boldsymbol{T_A}$ ***is one-to-one.***
- ***(l)*** $\boldsymbol{T_A}$ ***is onto.***

Section 6.4: Composition and invertibility of linear transformations

Theorem 6.4.1. *If* $T_1\colon \mathbb{R}^n \to \mathbb{R}^k$ *and* $T_2\colon \mathbb{R}^k \to \mathbb{R}^m$ *are linear transformations, then* $T_2 \circ T_1\colon \mathbb{R}^n \to \mathbb{R}^m$ *is also a linear transformation.*

Theorem 6.4.2. *Let* A *and* B *be matrices such that* BA *is defined. Then* $T_B \circ T_A = T_{BA}$.

Theorem 6.4.5. *If* T *is a one-to-one linear transformation, then so is* T^{-1}.

Theorem 6.4.6. *Let* $T = T_A$ *be a one-to-one linear operator with standard matrix* A. *Then* A *is invertible and* $T_A^{-1} = T_{A^{-1}}$.