Section 7.1: Basis and Dimension

Definition 7.1.1. A set of vectors in a subspace V of $\mathbb{R}^n$ is a **basis** for V if it is linearly independent and spans V.

Theorem 7.1.2. *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of nonzero vectors in $\mathbb{R}^n$. The S is linearly dependent if and only if some $\mathbf{v}_j$ is a linear combination of its predecessors.*

Theorem 7.1.3 (Existence of a Basis). *Every subspace of $\mathbb{R}^n$ has a basis (and this basis has at most n vectors).*

A basis for the subspace $\{\mathbf{0}\}$ is the empty set (despite what the book says).

Theorem 7.1.4. *All bases for a subspace V of $\mathbb{R}^n$ have the same number of vectors.*

Definition 7.1.5. The number of vectors in a basis is called the **dimension** of V.

The general solution to a homogeneous system that results from Gauss–Jordan elimination produces a basis for the solution space. This basis is called the **canonical basis**.

Section 7.2: Properties of Bases

Theorem 7.2.1. *If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a basis for a subspace V, then every vector in V can be expressed in* exactly one way *as a linear combination of the $\mathbf{v}_i$.*

Theorem 7.2.2. *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a subspace V.*

- (a) *If $V = \operatorname{span}\{S\}$, but S is not linearly independent, then a basis for V can be obtained by removing vectors from S.*
- (b) *If S is linearly independent, but $V \neq \operatorname{span}\{S\}$, then a basis for V can be obtained by adding vectors to S.*

Theorem 7.2.3. $\dim(V) =$ *the maximum number of linearly independent vectors in V*

Theorem 7.2.4. *Let V, W be subspaces of $\mathbb{R}^n$ such that V is contained in W.*

- (a) $0 \leq \dim(V) \leq \dim(W) \leq n$
- (b) *$V = W$ if and only if $\dim(V) = \dim(W)$.*

Theorem 7.2.6. *Let S be a set of l vectors in a subspace V with $\dim(V) = k$.*

- (a) *If $l = k$ and S is linearly independent, then S is a basis for V.*
- (b) *If $l = k$ and $V = \operatorname{span}(S)$, then S is a basis for V.*
- (c) *If $l < k$, then $V \neq \operatorname{span}(S)$.*
- (d) *If $l > k$, then S is not linearly independent.*

Theorem 7.2.7 (Unifying theorem). *Let A be an $n \times n$ matrix. TFAE:*

- (c) *A is invertible.*
- (k) *the columns of A are linearly independent.*
- (l) *the rows of A are linearly independent.*
- ***(m) the columns of A span $\mathbb{R}^n$.***
- ***(n) the rows of A span $\mathbb{R}^n$.***
- ***(o) the columns of A form a basis for $\mathbb{R}^n$.***
- ***(p) the rows of A form a basis for $\mathbb{R}^n$.***