Section 7.3: The fundamental spaces of a matrix

There are **four fundamental subspaces** associated to an $m \times n$ matrix A:

$$\begin{aligned}
\mathrm{row}(A) &= \text{the span of the row vectors of } A, && \text{a subspace of } \mathbb{R}^n \\
\mathrm{col}(A) &= \text{the span of the column vectors of } A, && \text{a subspace of } \mathbb{R}^m \\
\mathrm{null}(A) &= \text{the set of solutions to } A\mathbf{x} = \mathbf{0}, && \text{a subspace of } \mathbb{R}^n \\
\mathrm{null}(A^T) &= \text{the set of solutions to } A^T\mathbf{x} = \mathbf{0}, && \text{a subspace of } \mathbb{R}^m
\end{aligned}$$

Note that $\mathrm{row}(A^T) = \mathrm{col}(A)$ and $\mathrm{col}(A^T) = \mathrm{row}(A)$.

Definition 7.3.1. Let A be a matrix. The **rank** of A, denoted by $\mathrm{rank}(A)$, is the dimension of the row space of A. The **nullity** of A, denoted by $\mathrm{nullity}(A)$, is the dimension of the null space of A.

Definition 7.3.2. Let S be a non-empty set in $\mathbb{R}^n$. The **orthogonal complement** of S, denoted by $S^\perp$, is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in S.

Theorem 7.3.3. *Let S be a non-empty set in $\mathbb{R}^n$. Then $S^\perp$ is a subspace of $\mathbb{R}^n$.*

Theorem 7.3.4. *Let S be a non-empty set in $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$.*

- (a) $W^\perp \cap W = \{0\}$.
- (b) $S^\perp = \mathrm{span}(S)^\perp$.
- (c) $(W^\perp)^\perp = W$.

Theorem 7.3.5 & 7.3.6. *Let A be an $m \times n$-matrix. The row space and the null space of A are orthogonal complements. Similarly, the column space of A and the null space of A^T are orthogonal complements.*

Theorem 7.3.7. *Let A be a matrix.*

- (a) *Elementary row operations do not change the row space of A.*
- (b) *Elementary row operations do not change the null space of A.*
- (c) *The non-zero row vectors in any row echelon form of A form a basis for the row space of A.*

Section 7.4: The dimension theorem and its implications

Theorem 7.4.1 (Dimension theorem for matrices). *Let A be an $m \times n$-matrix. Then* $\mathrm{rank}(A) + \mathrm{nullity}(A) = n$.

Theorem 7.4.2. *Let A be an $m \times n$-matrix of rank k. Then:*

- (a) *A has nullity $n - k$.*
- (b) *Every row echelon form of A has k non-zero rows.*
- (c) *Every row echelon form of A has $m - k$ zero rows.*
- (d) *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has k pivot (leading) variables and $n - k$ free variables.*

Theorem 7.4.3 (Dimension theorem for subspaces). *Let W be a subspace of $\mathbb{R}^n$. Then* $\dim W + \dim W^\perp = n$.

Theorem 7.4.4 (Unifying theorem). *Let A be an $n \times n$ matrix. TFAE:*

(d) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(n) The row vectors of A span $\mathbb{R}^n$.

(q) $\mathbf{rank}(\boldsymbol{A}) = \boldsymbol{n}$.

(r) $\mathbf{nullity}(\boldsymbol{A}) = \mathbf{0}$.

Theorem 7.4.5. *Let W be a subspace of $\mathbb{R}^n$ of dimension $n-1$. Then there is a (non-zero) vector* $\mathbf{a}$ *such that $W = \mathbf{a}^{\perp}$; that is, W is a hyperplane through the origin.*

Section 7.5: The rank theorem and its implications

Theorem 7.5.1 (Rank theorem). *The row space and the column space of a matrix have the same dimension.*

Theorem 7.5.2. *For any matrix A one has* $\text{rank}(A) = \text{rank}(A^T)$.

Theorem 7.5.3. *Let $A\mathbf{x} = \mathbf{b}$ be a linear system of m equations in n unknowns. TFAE:*

(a) $A\mathbf{x} = \mathbf{b}$ *is consistent.*

(b) $\mathbf{b}$ *is in the column space of A.*

(c) *The augmented matrix $[A\ \mathbf{b}]$ has the same rank as A.*

Definition 7.5.4. A matrix is said to have **full column rank** if its column vectors are linearly independent, and **full row rank** if its row vectors are linearly independent.

Theorem 7.5.6. *Let A be an $m \times n$ matrix. TFAE:*

(a) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$ *in $\mathbb{R}^m$.*

(c) *A has full column rank.*

Theorem 7.5.7. *Let A be an $m \times n$ matrix. Then:*

(a) *(Overdetermined) If $m > n$, then there exists* $\mathbf{b}$ *in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent.*

(b) *(Underdetermined) If $m < n$, then for every* $\mathbf{b}$ *in $\mathbb{R}^m$, $A\mathbf{x} = \mathbf{b}$ is either inconsisten or has infinitely many solutions.*

Section 7.6: The Pivot Theorem and Its Implications

Theorem 7.6.1. *Let A and B be row equivalent matrices (that is, they are related by row operations). Then a subset of the columns of A is linearly independent if and only if the corresponding subset of the columns of B is linearly independent.*

Theorem 7.6.3 (The Pivot Theorem). *The pivot columns of a matrix form a basis for its column space.*

Finding bases for the four fundamental spaces of A: All four bases can be found using a single row-reduction procedure. Let U be a row echelon form of A and let R be the reduced row echelon form. Then bases are given by the following vectors:

1. $\mathbf{row}(\boldsymbol{A})$: the nonzero rows of U or R.
2. $\mathbf{col}(\boldsymbol{A})$: the pivot columns of A, identified using U or R.
3. $\mathbf{null}(\boldsymbol{A})$: the canonical solutions of $R\mathbf{x} = \mathbf{0}$.
4. $\mathbf{null}(\boldsymbol{A}^{\boldsymbol{T}})$: while row reducing A, also apply operations to the identity matrix; basis given by rows of resulting matrix which are beside the zero rows of R.