Definition 7.11.1. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be an ordered basis for a subspace W of $\mathbb{R}^n$ and let $\mathbf{w}$ be in W. If $\mathbf{w} = a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k$, then $a_1, \ldots, a_k$ are called the **coordinates of w with respect to *B***, $(\mathbf{w})_B = (a_1, \ldots, a_k)$ is called the **coordinate vector** for **w** with respect to B, and

$$[\mathbf{w}]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}$$

is called the **coordinate matrix** for **w** with respect to B.

Theorem 7.11.2. *Let B be an* orthonormal *basis for a k-dimensional subspace W of $\mathbb{R}^n$, and let $\mathbf{u}$ and $\mathbf{v}$ be vectors in W with coordinate vectors $(\mathbf{u})_B = (u_1, \ldots, u_k)$ and $(\mathbf{v})_B = (v_1, \ldots, v_k)$. Then*

(*a*) $\|\mathbf{u}\| = \sqrt{u_1^2 + \cdots + u_k^2}$,

(*b*) $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + \cdots + u_kv_k = (\mathbf{u})_B \cdot (\mathbf{v})_B$.

Theorem 7.11.3 (Change of basis). *Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $B' = \{\mathbf{v}'_1, \ldots, \mathbf{v}'_k\}$ be bases for $\mathbb{R}^n$. The coordinate matrices of $\mathbf{w} \in \mathbb{R}^n$ with respect to the two bases are related by the equation*

$$[\mathbf{w}]_{B'} = P_{B \to B'}[\mathbf{w}]_B,$$

where

$$P_{B \to B'} = \left[\, [\mathbf{v}_1]_{B'} \ \cdots \ [\mathbf{v}_n]_{B'} \,\right]$$

*is the **transition matrix** (or **change of coordinates matrix**) from B to B'.*

Theorem 7.11.4. *Let B and B' be bases for $\mathbb{R}^n$. The transition matrices $P_{B \to B'}$ and $P_{B' \to B}$ are invertible and inverses of one another, that is,*

$$P_{B' \to B} = (P_{B \to B'})^{-1}.$$

Theorem 7.11.5. *Let B be a basis for $\mathbb{R}^n$. The coordinate map $\mathbf{x} \mapsto (\mathbf{x})_B$ (or $\mathbf{x} \mapsto [\mathbf{x}]_B$) is a 1-1 and onto linear operator on $\mathbb{R}^n$. Moreover, if B is an orthonormal basis, then the coordinate map is an orthogonal operator.*

Theorem 7.11.7. *Let B and B' be orthonormal bases for $\mathbb{R}^n$. Then the transition matrices $P_{B \to B'}$ and $P_{B' \to B}$ are orthogonal.*

Theorem 7.11.8. *Let $P = [\,\mathbf{p}_1 \ldots \mathbf{p}_n\,]$ be an invertible $n \times n$ matrix. Then P is the transition matrix from the basis $B = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$ to the standard basis $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$.*