Section 8.1: Matrix representations of linear transformations

Theorem 8.1.1. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator and $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$. Define the $n \times n$ matrix*

$$[T]_B = \big[[T(\mathbf{v}_1)]_B \cdots [T(\mathbf{v}_n)]_B\big].$$

Then

$$[T(\mathbf{x})]_B = [T]_B[\mathbf{x}]_B$$

for all $\mathbf{x} \in \mathbb{R}^n$. Moreover, $[T]_B$ is the only matrix with this property. It is called the ***matrix for T with respect to B****.*

Theorem 8.1.2 and Theorem 8.1.3. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator and $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \dots, \mathbf{v}'_n\}$ be bases for $\mathbb{R}^n$. Then*

$$[T]_{B'} = P[T]_B P^{-1},$$

where $P = P_{B \to B'}$. If B and B' are orthonormal bases, then P is orthogonal, so that

$$[T]_{B'} = P[T]_B P^T.$$

This applies in particular to the special case where $B' = S$ is the standard basis for $\mathbb{R}^n$, in which case $[T]_{B'} = [T]$.

Theorem 8.1.4. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Define the $m \times n$ matrix*

$$A = \big[[T(\mathbf{v}_1)]_{B'} \cdots [T(\mathbf{v}_n)]_{B'}\big].$$

Then

$$[T(\mathbf{x})]_{B'} = A[\mathbf{x}]_B$$

for all $\mathbf{x} \in \mathbb{R}^n$. Moreover, A is the only matrix with this property. We write $[T]_{B',B} = A$.

Section 8.2: Similarity and diagonalizability

Definition 8.2.1. If A and C are square matrices with the same size, then we say that ***C* is similar to *A*** if there is an invertible matrix P such that $C = P^{-1}AP$.

Theorem 8.2.2. *Two square matrices are similar if and only if there exist bases with respect to which they represent the same linear operator.*

Theorem 8.2.3. *Similar matrices have the same determinant, rank, nullity, trace, characteristic polynomial and eigenvalues, and the eigenvalues have the same algebraic multiplicities.*

Definition. If λ_0 is an eigenvalue of A, the dimension of the eigenspace corresponding to λ_0 is called the **geometric multiplicity** of λ_0.

Theorem 8.2.4. *The eigenvalues of similar matrices have the same geometric multiplicities.*

Theorem 8.2.5. *Suppose that $C = P^{-1}AP$ and that λ is an eigenvalue of A and C.*

- *(a) If $\mathbf{x}$ is an eigenvector of C corresponding to λ, then $P\mathbf{x}$ is an eigenvector of A corresponding to λ.*
- *(b) If $\mathbf{x}$ is an eigenvector of A corresponding to λ, then $P^{-1}\mathbf{x}$ is an eigenvector of C corresponding to λ.*

The diagonalization problem. Given a square matrix A, does there exist an invertible matrix P for which $P^{-1}AP$ is a diagonal matrix? If so, how do we find P? If such a P exists, A is said to be **diagonalizable** and P is said to **diagonalize** A.

Theorem 8.2.6. *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. (These vectors must then form a basis for $\mathbb{R}^n$.)*

Method for diagonalizing A:

Step 1. Find n linearly independent eigenvectors of A, say $\mathbf{p}_1, \ldots, \mathbf{p}_n$.

Step 2. Form the matrix $P = [\mathbf{p}_1, \ldots, \mathbf{p}_n]$.

Step 3. The matrix $P^{-1}AP$ will be diagonal and will have the eigenvalues corresponding to $\mathbf{p}_1, \ldots, \mathbf{p}_n$ (in this order) on the diagonal.

Theorem 8.2.7. *If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors of A corresponding to* distinct *eigenvalues, then they are linearly independent.*

Theorem 8.2.8. *An $n \times n$ matrix with n distinct real eigenvalues is diagonalizable.*

Note that a matrix can still be diagonalizable even if it does not have n distinct eigenvalues!

Theorem 8.2.9. *An $n \times n$ matrix is diagonalizable if and only if the sum of the geometric multiplicities is n.*

Theorem 8.2.10. *If A is a square matrix, then:*

(*a*) *For each eigenvalue λ of A,*

$$\text{geometric multiplicity of } \lambda \leq \text{algebraic multiplicity of } \lambda.$$

(*b*) *A is diagonalizable if and only if for every eigenvalue λ of A,*

$$\text{geometric multiplicity of } \lambda = \text{algebraic multiplicity of } \lambda.$$

Theorem 8.2.11. *If A is an $n \times n$ matrix, then the following are equivalent:*

(*a*) *A is diagonalizable.*

(*b*) *A has n linearly independent eigenvectors.*

(*c*) *$\mathbb{R}^n$ has a basis consisting of eigenvectors of A.*

(*d*) *The sum of the geometric multiplicities of the eigenvalues of A is n.*

(*e*) *The geometric multiplicity of each eigenvalue of A is the same as the algebraic multiplicity.*