(16 pts) **1.** For each of the following, circle **T** if the statement is always true and circle **F** if it can be false.
If you are unsure, leave blank. Wrong answers will receive **−2 marks**.
We justify the answers to help the reader.

(a) If A is a 3×4 matrix and $\mathbf{b}$ is in $\mathbb{R}^3$,
then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. **F**
The system may have no solutions.

(b) If A is an invertible matrix, then the system $A\mathbf{x} = \mathbf{b}$
has exactly one solution for every $\mathbf{b}$. **T**
This is part of the unifying theorem.

(c) For any $m \times n$ matrix A, the set of solutions to $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$. **T**
This is a theorem we covered.

(d) Every set of four vectors in $\mathbb{R}^3$ is linearly dependent. **T**
At most three vectors in $\mathbb{R}^3$ can be linearly independent.

(e) If A is a square matrix with two identical columns, then $\det(A) = 0$. **T**
This is a theorem we covered.

(f) If A is skew-symmetric, then A^T is symmetric. **F**
A^T is also skew-symmetric.

(g) $\lambda = 1$ is an eigenvalue of every square matrix A. **F**
It is not an eigenvalue of the zero matrix, for instance.

(h) If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of A with eigenvalue 2,
then $\mathbf{u} + \mathbf{v}$ is either $\mathbf{0}$ or is also an eigenvector with eigenvalue 2. **T**
Solution: $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = 2\mathbf{u} + 2\mathbf{v} = 2(\mathbf{u} + \mathbf{v})$

For all remaining problems, you must SHOW ALL OF YOUR WORK and EXPLAIN FULLY and CLEARLY to receive full credit.

(3 pts) **2.** Compute the volume of the parallelepiped formed by the vectors $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 13 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 25 \\ 1 \end{bmatrix}$.

The volume V of the parallelepiped spanned by three vectors in $\mathbb{R}^3$ is the absolute value of the determinant of the matrix which has the given vectors as columns. Here we have

$$\begin{vmatrix} 2 & 13 & -3 \\ 0 & -1 & 25 \\ 0 & 0 & 1 \end{vmatrix} = 2 \cdot (-1) \cdot 1 = -6$$

since the matrix is upper triangular. Hence, $V = |-6| = 6$.

3. Let $\mathbf{u} = (3, 4, 5)$ and $\mathbf{v} = (-4, 3, 5)$.

(4 pts) (a) Evaluate $\|\mathbf{u}\|$ and $\mathbf{u} - 2\mathbf{v}$.

$$\|\mathbf{u}\| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}.$$

$$\mathbf{u} - 2\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - 2 \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \\ -5 \end{bmatrix}.$$

(2 pts) (b) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \cdot \mathbf{v} = 3 \cdot (-4) + 4 \cdot 3 + 5 \cdot 5 = 25,$$
$$\|\mathbf{v}\| = \sqrt{(-4)^2 + 3^2 + 5^2} = \sqrt{50}.$$

Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\arccos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{25}{(\sqrt{50})^2} = \frac{1}{2},$$

hence $\theta = \pi/3$, which is 60 degrees.

(2 pts) (c) Find a non-zero vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

The crossproduct gives a vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \times \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \cdot 5 - 5 \cdot 3 \\ 5 \cdot (-4) - 3 \cdot 5 \\ 3 \cdot 3 - 4 \cdot (-4) \end{bmatrix} = \begin{bmatrix} 5 \\ -35 \\ 25 \end{bmatrix},$$

which is non-zero.

(3 pts) **4.** (a) Find parametric equations of the plane given by $3x - y + 2z = -1$.

We write the equation of the plane as $y = 3x + 2z + 1$ and introduce parameters $s = x$ and $t = z$. Then

$$x = s,$$
$$y = 1 + 3s + 2t,$$
$$z = t.$$

(3 pts) (b) Do the planes $3x - y + 2z = -1$ and $2x - y + 2z = 0$ intersect? Explain.

The given normal vectors of the two planes are not parallel, so the planes must intersect.

(3 pts) **5.** Find the area of the triangle with vertices $\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

The sides of the triangle meeting at $(2, 1, 1)$ are given by the vectors $\mathbf{u} = (2, 4, 1) - (2, 1, 1) = (0, 3, 0)$ and $\mathbf{v} = (0, 1, 2) - (2, 1, 1) = (-2, 0, 1)$. The area of the triangle is given by

$$A = \frac{1}{2} \|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2} \left\| \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\| = \frac{1}{2} \left\| \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \right\| = \frac{3}{2} \left\| \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\| = \frac{3}{2}\sqrt{5}.$$

6. Let $A = \begin{bmatrix} -3 & 1 & -1 \\ 0 & -3 & 0 \\ 0 & -2 & 2 \end{bmatrix}$.

(3 pts) (a) Find the characteristic polynomial of A.

The characteristic polynomial $p_A(\lambda)$ of A is the determinant of $\lambda I - A$. Using cofactor expansion in the first column we get:

$$p_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda+3 & -1 & 1 \\ 0 & \lambda+3 & 0 \\ 0 & 2 & \lambda-2 \end{vmatrix} = (\lambda+3)\begin{vmatrix} \lambda+3 & 0 \\ 2 & \lambda-2 \end{vmatrix} = (\lambda+3)^2(\lambda-2)$$

(2 pts) (b) Find the eigenvalues of A and list their algebraic multiplicities.

The eigenvalues of A are the zeroes of $p_A(\lambda)$. Hence, they are $\lambda = -3$ with algebraic multiplicity 2 and $\lambda = 2$ with algebraic multiplicity 1.

(4 pts) (c) Find the eigenspace associated to the *largest* eigenvalue you found.

The largest eigenvalue is $\lambda = 2$. We need to solve the system $(2I - A)\mathbf{x} = \mathbf{0}$. The matrix $2I - A$ is

$$\begin{bmatrix} 5 & -1 & 1 \\ 0 & 5 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Row reducing this (work needs to be shown) leads to

$$\begin{bmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which has general solution

$$x_1 = -\frac{1}{5}s,$$
$$x_2 = 0,$$
$$x_3 = s.$$

So the eigenspace is the span of $\begin{bmatrix} -\frac{1}{5} \\ 0 \\ 1 \end{bmatrix}$.

7. Suppose that A and B are 3×3 matrices with $\det(A) = 2$ and $\det(B) = -3$. Compute the following. *No justification needed, no part marks.*
We justify the answers to help the reader.

(2 pts) (a) $\det(2B^T) =$ ____________

$$\det(2B^T) = 2^3 \det B^T = 8 \det B = -24$$

(2 pts) (b) $\det(AB^2A^{-1}) =$ ____________

$$\det(AB^2A^{-1}) = (\det A)(\det B)^2(\det A)^{-1} = (\det B)^2 = 9$$

8. Suppose that C is a 4×4 matrix with eigenvalues -1, 1, 2 and 3. Compute the following. *No justification needed.*

(2 pts) (a) $\operatorname{tr}(C) =$ ____________

$$\operatorname{tr} C = -1 + 1 + 2 + 3 = 5$$

(2 pts) (b) $\det(C) =$ ____________

$$\det C = (-1) \cdot 1 \cdot 2 \cdot 3 = -6$$

(2 pts) (c) $\det(2I_4 - C) =$ ____________

$$\det(2I_4 - C) = 0 \quad \textit{because 2 is an eigenvalue of } C$$

(2 pts) (d) $\operatorname{tr}(2I_4 - C) =$ ____________

$$\operatorname{tr}(2I_4 - C) = 2 \operatorname{tr} I_4 - \operatorname{tr} C = 8 - 5 = 3$$

(3 pts) **9.** Compute the matrix product

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 11 \\ 1 & -2 \end{bmatrix}$$

10. Let $\mathbf{v}_1 = \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$.

(4 pts) (a) Find scalars c_1, c_2, c_3 and c_4 (not all zero) such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}\,.$$

Putting the vectors into the columns of a matrix, we need to solve the system

$$\begin{bmatrix} -2 & 1 & -5 & 0 \\ -3 & -2 & 3 & 0 \\ -4 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \mathbf{0}.$$

Row reducing the coefficient matrix (work needs to be shown) leads to

$$\begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -1/2 \end{bmatrix},$$

which has general solution

$$\begin{aligned} c_1 &= -s/2, \\ c_2 &= 3s/2, \\ c_3 &= s/2, \\ c_4 &= s. \end{aligned}$$

Taking $s = 2$ (for example) tells us that

$$-\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3 + 2\mathbf{v}_4 = \mathbf{0}$$

(2 pts) (b) Write one of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ as a linear combination of the remaining ones.

We simply solve for any of the vectors with a non-zero coefficient. For example,

$$\mathbf{v}_1 = 3\mathbf{v}_2 + \mathbf{v}_3 + 2\mathbf{v}_4$$

(4 pts) **11.** Compute the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & -2 \\ 1 & -2 & 0 \end{bmatrix}$.

Row reducing

$$\begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ -2 & 1 & -2 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(work needs to be shown) leads to

$$\begin{bmatrix} 1 & 0 & 0 & -4/3 & -2/3 & 1 \\ 0 & 1 & 0 & -2/3 & -1/3 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}.$$

So the inverse is

$$A^{-1} = \begin{bmatrix} -4/3 & -2/3 & 1 \\ -2/3 & -1/3 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

For this question, and many others, it is easy to check whether your answer is correct. (Note in particular that A^{-1} must be symmetric because A is.)