Linear Algebra 040b Final Examination Monday, April 25, 2005

1. [*8 marks*] Find a basis for the null space of A, where

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{bmatrix}.$$

2. [*8 marks*] Suppose that

$$B = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 1 & 3 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix},$$

and that the 3×3 matrix X satisfies $B(X + C) = D$. Find X.

3. [*8 marks*] Let

$$E = \begin{bmatrix} -1 & 7 & 8 \\ 0 & 5 & 6 \\ 0 & 0 & 4 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 3 & 0 & 0 \\ -1 & -5 & 0 \\ 0 & 1 & -2 \end{bmatrix}, \quad .$$

Find $\det(E)$, $\det(F)$, $\det(EF)$, and $\det(E + F)$.

4. [*8 marks*] Find the standard matrix of the projection of $\mathbf{R}^3$ on to the subspace spanned by $(1, 1, -2)$ and $(1, 0, -1)$.

5. Let ℓ be the line in $\mathbf{R}^3$ passing through the points $(2, 4, 1)$ and $(4, 0, 7)$.
   (a) [*2 marks*] Find parametric equations for ℓ.
   (b) [*2 marks*] Find the point of intersection of ℓ with the xy-plane.
   (c) [*4 marks*] Find the point on ℓ that is closest to $(8, -5, 7)$.

6. The characteristic polynomial of

$$G = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}.$$

is

$$\det(\lambda I - G) = (\lambda - 1)(\lambda + 3)^2.$$

   (a) [*4 marks*] Find a basis for the eigenspace of G corresponding to the eigenvalue $\lambda = 1$.
   (b) [*4 marks*] Find a basis for the eigenspace of G corresponding to the eigenvalue $\lambda = -3$.
   (c) [*2 marks*] Is G diagonalizable? Justify your answer.

7. [*8 marks*] Find the solution of $H\mathbf{x} = \mathbf{b}$ that lies in the row space of H, where

$$H = \begin{bmatrix} 1 & -1 & 0 & 2 & 0 \\ 3 & 1 & 2 & 1 & 1 \\ -1 & 0 & 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 9 \\ 5 \\ 9 \end{bmatrix}.$$

8. [*6 marks*] Find an invertible matrix P and a diagonal matrix Q such that

$$P^{-1}QP = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

9. Let $\mathcal{B}$ and $\mathcal{B}'$ be the bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{B}' = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

(a) [*4 marks*] Find $P_{\mathcal{B}\to\mathcal{B}'}$, the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.
(b) [*2 marks*] Is $P_{\mathcal{B}\to\mathcal{B}'}$ orthogonal? Justify your answer.
(c) [*2 marks*] If $[x]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$ find $[x]_{\mathcal{B}'}$.
(d) [*2 marks*] If $[x]_{\mathcal{B}'} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ find $[x]_{\mathcal{B}}$.

10. [*6 marks*] Let $T : \mathbf{R}^3 \to \mathbf{R}^4$ be defined by

$$T(x, y, z) = (x + 3z, 3z - y, 3z + 4y - x, x - 2y + 3z).$$

Find vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ such that $\{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3)\}$ is an orthogonal set in $\mathbf{R}^4$.

11. Circle the correct answers.

(a) [*2 marks*] If $\mathbf{u} = (-2, 1, 4)$ and $\mathbf{v} = (1, 3, -2, -1)$ are row vectors then the matrix $\mathbf{u}^T\mathbf{v}$ has rank 1. **T** **F**

(b) [*2 marks*] If A is the standard matrix of the linear transformation S then $\ker(S) = \text{null}(A)$ **T** **F**

(c) [*2 marks*] If A is an orthogonal matrix then $\det(A) = 1$. **T** **F**

(d) [*2 marks*] If U and V are $n \times n$ matrices with $\det(U) = 2$ and $\det(V) = 5$ then $\det(U^3V^{-1}U^TV) = 400$. **T** **F**

(e) [*2 marks*] The system of equations

$$\begin{aligned} x + 3y &= 5 \\ -2x + 3y + z &= 1 \\ y + z &= -2. \end{aligned}$$

can be solved using Cramer's rule. **T** **F**

(f) [*2 marks*] For any matrix B, $\text{rank}(B) = \text{rank}(B^TB) = \text{rank}(BB^T)$. **T** **F**

(g) [*2 marks*] $\{(1, 1, 2), (1, 0, 1), (2, 1, 3)\}$ is a basis of $\mathbf{R}^3$. **T** **F**

(h) [*2 marks*] If

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

then the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. **T** **F**

(i) [*2 marks*] The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has distinct, real eigenvalues if and only if $(a - d)^2 + 4bc > 0$. **T** **F**

(j) [*2 marks*] For any $m \times n$ matrix A and any $m \times 1$ column vector $\mathbf{b}$ the system $(A^TA)\mathbf{x} = A^T\mathbf{b}$ is consistent. **T** **F**