Linear Algebra 1600b Midterm *2:00-5:00 pm* Febuary 28, 2009

Last Name ______________________________

First Name ______________________________

Student ID ______________________________

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14	3	7	7	6	4	6	6	5	4	8	70

This exam has 11 problems on 10 pages.

CIRCLE LECTURE AND LAB SECTIONS:

LECTURE:

001 MWF 12:30	002 MWF 11:30

LAB:

003 Th 1:30	004 Th 12:30	005 Th 2:30	006 Th 10:30	007 W 10:30	008 W 9:30

NO CALCULATORS, NOTES OR OTHER AIDS.

(14 pts) **1.** For each of the following, circle **T** if the statement is always true and circle **F** if it can be false. **Do not guess**: wrong answers will receive **−2 marks**.

(a) If A is a $n \times n$ matrix, then $A - A^T$ is skew-symmetric. **T** **F**
Solution: True: $(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)$.

(b) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$ that span a plane, then $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. **T** **F**
Solution: False: rather, they must be linearly independent.

(c) If A is an invertible matrix, then the system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$. **T** **F**
Solution: True: solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

(d) If A is an invertible matrix and two rows are interchanged, then the resulting matrix is invertible. **T** **F**
Solution: True: exchanging rows changes determinant by a sign.

(e) If A and B are $n \times n$ matrices, then $\det(A + B) = \det(A) + \det(B)$. **T** **F**
Solution: False: however, $\det(AB) = \det(A)\det(B)$.

(f) If $\lambda = 0$ is an eigenvalue of A, then A is invertible. **T** **F**
Solution: False: if 0 is an eigenvalue, determinant is zero.

(g) If A is a 3×4 matrix, then the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions. **T** **F**
Solution: True: after row-reducing, at most three leading ones, but four columns.

For all remaining problems, you must SHOW ALL OF YOUR WORK and EXPLAIN FULLY to receive full credit.

(3 pts) **2.** Find the area of the triangle with vertices at $(1, 1)$, $(4, 6)$ and $(0, 3)$.

Solution: Let $\mathbf{v} = (4 - 1, 6 - 1)$ and $\mathbf{u} = (0 - 1, 3 - 1)$ be the vectors along the two sides of the triangle that meet at the point $(1, 1)$. Then $\mathbf{v} = (3, 5)$ and $\mathbf{u} = (-1, 2)$. The triangle has half the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$, so its area is

$$\begin{aligned} &= \frac{1}{2}\left|\det\begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}\right| \\ &= (6 + 5)/2 \\ &= 11/2. \end{aligned}$$

3. Let $\mathbf{u} = (3, 2, 1)$ and $\mathbf{v} = (1, 2, 3)$.

(4 pts) (a) Evaluate $\|\mathbf{u}\|$, $3\mathbf{u} - \mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$.

Solution:

$$\begin{aligned}
\|\mathbf{u}\| &= \sqrt{3^2+2^2+1^2} \\
&= \sqrt{14}; \\
3\mathbf{u}-\mathbf{v} &= 3(3,2,1)-(1,2,3) \\
&= (8,4,0); \\
\mathbf{u}\cdot\mathbf{v} &= 3\cdot 1+2\cdot 2+1\cdot 3 \\
&= 10; \\
\mathbf{u}\times\mathbf{v} &= (2\cdot 3-2\cdot 1, -(3\cdot 3-1\cdot 1), (3\cdot 2-2\cdot 1)) \\
&= (4,-8,4)
\end{aligned}$$

(3 pts) (b) Find all scalars s and t for which $s\mathbf{u}+t\mathbf{v}$ is orthogonal to the vector $(1,-1,1)$.

Solution: The vector $s\mathbf{u}+t\mathbf{v}$ is orthogonal to $(1,-1,1)$ if and only if $(3s+t, 2s+2t, s+3t)\cdot(1,-1,1)=0$. That is,

$$\begin{aligned}
0 &= 3s+t-(2s+2t)+s+3t \\
&= 2s+2t,
\end{aligned}$$

so the vectors are orthogonal for all scalars t and $s=-t$.

(4 pts) **4.** (a) Use row operations to put the following matrix into reduced row echelon form. You must show the intermediate matrices, but you don't have to describe the row operations you used.

$$A = \begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 1 & -1 & 5 & -1 & 2 \\ -2 & 1 & -8 & 1 & -3 \end{bmatrix}$$

Solution: Row-reduce to

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & -1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 1 & -1 \end{bmatrix},$$

then

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & -1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

then

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(3 pts) (b) Using the previous part, write down the general solution to the following system in parametric form.

$$\begin{aligned}
x+3z &= 1 \\
x-y+5z-w &= 2 \\
-2x+y-8z+w &= -3
\end{aligned}$$

Solution: The calculation above row-reduced the augmented matrix for this system. So we see that z and w are free variables: let $z=s$ and $w=t$, so that the general solution is

$$\begin{aligned}
x &= 1-3s \\
y &= -1+2s-t \\
z &= s \\
w &= t.
\end{aligned}$$

5. Dan was solving a system of equations in x, y and z. He found the reduced row-echelon form of his augmented matrix was

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

(3 pts) (a) He lost a page of his notes and could not remember what the original system of equations was. However, he remembers that he used only two row operations: first he subtracted row one from row three, and then he added three times row two to row three. What was the original system of equations?

Solution: We can undo the operations by first subtracting three times row 2 from row 3 to get

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -2 \\ 0 & -3 & -6 & 6 \end{array}\right].$$

Then add row 1 to row 3:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -2 \\ 1 & -3 & -7 & 7 \end{array}\right].$$

This corresponds to the system of equations

$$\begin{aligned} x \quad - z &= 1 \\ y + 2z &= -2 \\ x - 3y - 7z &= 7. \end{aligned}$$

(3 pts) (b) Does this system of equations have any solutions? If so, state how many solutions it has and say whether the solutions form a point, a line, a plane or all of R^3.

Solution: Yes: the row-reduced form had a full row of zeros, so the system is consistent. There is one free variable, so the solutions from a line in $\mathbb{R}^3$.

(4 pts) **6.** Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

Solution: Row-reduce $(A|I_3)$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 6 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1/2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1/2 & 0 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1/2 & 1 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & 1 & -1/2 & 1 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -1 & 1 & -2 \\ 1 & -1/2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$.

(3 pts) **7.** (a) Does the set of all vectors of the form $(0, a, b)$ with a and b real numbers form a subspace of $\mathbb{R}^3$? Explain fully.

Solution: Yes. Since $(0, 0, 0)$ has this form, the set is nonempty. Now check: if $\mathbf{v} = (0, a, b)$ and $\mathbf{w} = (0, a', b')$, then $\mathbf{v} + \mathbf{w} = (0, a + a', b + b')$, which is again a vector of this form.

Second, check that, for any vector $\mathbf{v}$ of the form $\mathbf{v} = (0, a, b)$ and for any scalar c, $c\mathbf{v} = (0, ca, cb)$ also has a zero in its first coordinate, so it has the same form.

(3 pts) (b) Does the set of all vectors of the form (a, b, c) with a, b and c real numbers with $a \geq 0$ form a subspace of $\mathbb{R}^3$? Explain fully.

Solution: No. In particular, $(1, 1, 1)$ is in this set, but $-(1, 1, 1) = (-1, -1, -1)$ is not, so it can't be a subspace.

Note that to show that something is false, you give an explicit counterexample.

(4 pts) **8.** (a) Use a determinant to find all values of c for which the vectors $\begin{bmatrix} 2 \\ 0 \\ c \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ are linearly independent.

Solution: Since

$$\begin{aligned} \det \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 3 \\ c & 0 & 1 \end{bmatrix} &= 2 \det \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix} + c \det \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \\ &= -2 + 2c, \end{aligned}$$

the determinant of the matrix with the given vectors as columns is nonzero as long as $c \neq 1$. So the vectors are linearly independent if and only if $c \neq 1$.

(2 pts) (b) Compute the volume of the parallelepiped formed by the vectors $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 13 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 25 \\ 1 \end{bmatrix}$.

Solution: We know that its volume is the absolute value of the determinant of the matrix with these three columns. The matrix is upper-triangular, so its determinant is the product of the diagonal entries, $2(-1)(1)$. So the volume is $|-2| = 2$.

9. Suppose that A and B are 3×3 matrices with $\det(A) = 3$ and $\det(B) = 2$. Compute the following. *No justification needed.*

(1 pts) (a) $\det(A^{-1}) = \det(A)^{-1} = 1/3$

(1 pts) (b) $\det(2A) = 2^3 \det(A) = 24$

(1 pts) (c) $\det(ABA) = \det A \det B \det A = 18$

(1 pts) (d) $\det(B^T) = \det B = 2$

(1 pts) (e) $\det(B^3) = \det B^2 = 8$

10. Suppose that C is a 4×4 matrix with eigenvalues 1, 2, 3 and 4. Compute the following. *No justification needed.*

(1 pts) (a) $\mathrm{tr}(C) = 1 + 2 + 3 + 4 = 10$

(1 pts) (b) $\det(C) = 1 \cdot 2 \cdot 3 \cdot 4 = 24$

(1 pts) (c) $\det(2I_4 - C) = 0$ since 2 is an eigenvalue

(1 pts) (d) $\mathrm{tr}(2I_4 - C) = \mathrm{tr}(2I_4) - \mathrm{tr}(C) = -2$

11. Let $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}$.

(3 pts) (a) Find the characteristic polynomial of A.

Solution: The characteristic polynomial is the determinant of $\lambda I_3 - A$, which is

$$\begin{aligned} \det \begin{bmatrix} \lambda - 1 & 1 & 1 \\ 0 & \lambda - 2 & 0 \\ 0 & -3 & \lambda - 2 \end{bmatrix} &= (\lambda - 1) \det \begin{bmatrix} \lambda - 2 & 0 \\ -3 & \lambda - 2 \end{bmatrix} \\ &= (\lambda - 1)(\lambda - 2)^2. \end{aligned}$$

(2 pts) (b) Find the eigenvalues of A and their algebraic multiplicities.

Solution: The eigenvalues are the roots of the characteristic polynomial. From above, those are $\lambda = 1$ (with multiplicity 1) and $\lambda = 2$ (with multiplicity two.)

(3 pts) (c) Find the eigenspace associated to the *smallest* eigenvalue you found.

Solution: The least eigenvalue is $\lambda = 1$, and its eigenspace is the set of solutions to $(1 \cdot I_3 - A)\mathbf{x} = \mathbf{0}$. So we row reduce $(I - A)$:

$$\begin{aligned} I - A &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & -3 & -1 \end{bmatrix} \\ &\to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & -1 \end{bmatrix} \\ &\to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

So the first variable is a free variable, and the general solution has the form $\mathbf{x} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ where t is an arbitrary scalar. That is, the eigenspace for the eigenvalue $\lambda = 1$ is the span of the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.