Linear Algebra 040b Midterm Examination Saturday, March 11, 2006

1. [*2 marks*] Find the sum of the vectors $(-3, 1, 2, 4)$ and $(4, 8, 1, 1)$.
2. [*2 marks*] Let θ be the angle between the vectors $(1, -2, 3)$ and $(3, 3, 1)$. Find $\cos\theta$.
3. [*2 marks*] Find the value of $c \in \mathbf{R}$ that makes the vectors $(2, 3, 1, 1)$ and $(-2, -1, c, -2)$ orthogonal.
4. [*2 marks*] Find a non-zero vector orthogonal to both $(4, 1, 0)$ and $(2, -1, 3)$.
5. [*6 marks*] Solve the following system of equations.

$$\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ x_1 + 3x_2 + 2x_3 &= 3 \\ x_2 + 2x_3 &= 4 \end{aligned}$$

6. [*6 marks*] Let

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find the matrices $I - A$, A^2, A^3 and $(I - A)^{-1}$.

7. Let

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 8 \\ 0 & 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 3 \\ 2 & 6 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 3 \\ 0 & 0 & -6 \end{bmatrix}.$$

   (a) [*4 marks*] Find elementary matrices E and F so that $EA = B$ and $FEA = C$.
   (b) [*2 marks*] Evaluate EFA.

8. [*6 marks*] Find the inverse of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

9. [*6 marks*] Let

$$M = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -1 & 1 & 4 \end{bmatrix}.$$

Find M_{23}, the $(2, 3)$-minor of M, and C_{23}, the $(2, 3)$-cofactor of M.

10. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ by defined by $T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, x_2 + x_3, 0)$.
   (a) [*2 marks*] Write down the standard matrix for T.
   (b) [*4 marks*] Find all vectors $\mathbf{x} \in \mathbf{R}^3$ such that $T(\mathbf{x}) = (8, 9, 0)$.
11. Let a be a real number and consider the system of equations

$$\begin{bmatrix} a & 2 & -1 \\ 3 & 1 & a \\ a & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

   (a) [*2 marks*] For what value or values of a does the system have a unique solution?
   (b) [*6 marks*] Use Cramer's rule to find this unique solution in terms of a.

12. [*6 marks*] Write down all the eigenvalues of the matrix

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

and find an eigenvector for each eigenvalue.

13. [*6 marks*] Find the null space of the matrix

$$C = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}.$$

Express your answer as the span of some set of vectors.

14. [*4 marks*] Let θ be a fixed angle and let G be the 4×4 matrix given in block form by

$$G = \begin{bmatrix} R_\theta & Z \\ Z & H_\theta \end{bmatrix},$$

where $Z = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is the 2×2 zero matrix, R_θ is the 2×2 rotation matrix, and H_θ is the 2×2 reflection matrix. Show that G is orthogonal and find the inverse of G.

15. Consider the vectors

$$\mathbf{v}_1 = (-1, 3, 1),\ \mathbf{v}_2 = (2, -3, 0),\ \mathbf{v}_3(1, 0, 0),\ \mathbf{v}_4 = (-1, 0, 4).$$

   (a) [*4 marks*] Is $\mathbf{v}_4$ in span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Justify your answer.
   (b) [*4 marks*] Is $\mathbf{v}_4$ in span$\{\mathbf{v}_1, \mathbf{v}_2\}$? Justify your answer.

16. [*4 marks*] Let $\mathbf{a} = (-2, 1, 2)$ and $\mathbf{u} = (4, -1, 0)$. Prove $S = \{\mathbf{u} + 3\mathbf{x} : \mathbf{a} \cdot \mathbf{x} = 3\}$ is a subspace of $\mathbf{R}^3$.

17. Circle the correct answers.
   (a) [*2 marks*] If A is a square matrix and AA^T is invertible, then so is A. **T** **F**
   (b) [*2 marks*] If B is an $m \times n$ matrix for which $B\mathbf{x} = \mathbf{0}$ has only the trivial solution, then the system $B^T\mathbf{y} = \mathbf{0}$ has only the trivial solution. **T** **F**
   (c) [*2 marks*] If three vectors form a linearly dependent set in $\mathbf{R}^3$ then each vector in the set can be expressed as a linear combination of the other two. **T** **F**
   (d) [*2 marks*] Every invertible matrix can be factored into a product of elementary matrices. **T** **F**
   (e) [*2 marks*] The eigenvalues of a square matrix C are the same as the eigenvalues of the reduced row echelon form of C. **T** **F**
   (f) [*2 marks*] The inverse of an orthogonal matrix is orthogonal. **T** **F**
   (g) [*2 marks*] If the linear operator $T : \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one and $T(\mathbf{u} - \mathbf{v}) = \mathbf{0}$ then $\mathbf{u} = \mathbf{v}$. **T** **F**
   (h) [*2 marks*] If G and H are $n \times n$ then $\det(G + H) = \det(G) + \det(H)$. **T** **F**
   (i) [*2 marks*] If D and E are $n \times n$ symmetric matrices, then $(DE)^T = DE$. **T** **F**
   (j) [*2 marks*] If $F\mathbf{x} = \mathbf{b}$ has infinitely many solutions, then so does $F\mathbf{x} = \mathbf{0}$. **T** **F**