Repeated multiplication can be handled similarly. The idea is to use the addition and multiplication tables to reduce the result of each calculation to 0, 1, or 2.

Extending these ideas to vectors is straightforward.

Example 1.14

In $\mathbb{Z}_3^5$, let $\mathbf{u} = [2, 2, 0, 1, 2]$ and $\mathbf{v} = [1, 2, 2, 2, 1]$. Then

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 2\cdot 1 + 2\cdot 2 + 0\cdot 2 + 1\cdot 2 + 2\cdot 1\\ &= 2 + 1 + 0 + 2 + 2\\ &= 1\end{aligned}$$

Vectors in $\mathbb{Z}_3^5$ are referred to as ***ternary vectors of length* 5.**

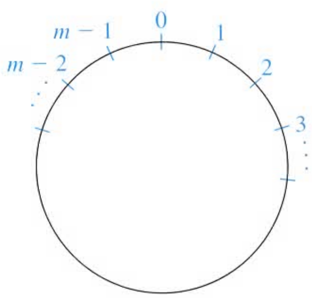

Figure 1.22
Arithmetic modulo m

In general, we have the set $\mathbb{Z}_m = \{0, 1, 2, \dots, m-1\}$ of ***integers modulo m*** (corresponding to an m-hour clock, as shown in Figure 1.22). A vector of length n whose entries are in $\mathbb{Z}_m$ is called an ***m-ary vector of length n***. The set of all m-ary vectors of length n is denoted by $\mathbb{Z}_m^n$.

Exercises 1.1

1. Draw the following vectors in standard position in $\mathbb{R}^2$:

(a) $\mathbf{a} = \begin{bmatrix}3\\0\end{bmatrix}$ (b) $\mathbf{b} = \begin{bmatrix}2\\3\end{bmatrix}$

(c) $\mathbf{c} = \begin{bmatrix}-2\\3\end{bmatrix}$ (d) $\mathbf{d} = \begin{bmatrix}3\\-2\end{bmatrix}$

2. Draw the vectors in Exercise 1 with their tails at the point $(1, -3)$.

3. Draw the following vectors in standard position in $\mathbb{R}^3$:

(a) $\mathbf{a} = [0, 2, 0]$ (b) $\mathbf{b} = [3, 2, 1]$
(c) $\mathbf{c} = [1, -2, 1]$ (d) $\mathbf{d} = [-1, -1, -2]$

4. If the vectors in Exercise 3 are translated so that their heads are at the point $(1, 2, 3)$, find the points that correspond to their tails.

5. For each of the following pairs of points, draw the vector $\overrightarrow{AB}$. Then compute and redraw $\overrightarrow{AB}$ as a vector in standard position.

(a) $A = (1, -1), B = (4, 2)$
(b) $A = (0, -2), B = (2, -1)$
(c) $A = (2, \frac{3}{2}), B = (\frac{1}{2}, 3)$
(d) $A = (\frac{1}{3}, \frac{1}{3}), B = (\frac{1}{6}, \frac{1}{2})$

6. A hiker walks 4 km north and then 5 km northeast. Draw displacement vectors representing the hiker's trip and draw a vector that represents the hiker's net displacement from the starting point.

Exercises 7–10 refer to the vectors in Exercise 1. Compute the indicated vectors and also show how the results can be obtained geometrically.

7. $\mathbf{a} + \mathbf{b}$ **8.** $\mathbf{b} + \mathbf{c}$

9. $\mathbf{d} - \mathbf{c}$ **10.** $\mathbf{a} - \mathbf{d}$

Exercises 11 and 12 refer to the vectors in Exercise 3. Compute the indicated vectors.

11. $2\mathbf{a} + 3\mathbf{c}$ **12.** $2\mathbf{c} - 3\mathbf{b} - \mathbf{d}$

13. Find the components of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are as shown in Figure 1.23.

14. In Figure 1.24, A, B, C, D, E, and F are the vertices of a regular hexagon centered at the origin.

Express each of the following vectors in terms of $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$:

(a) $\overrightarrow{AB}$ (b) $\overrightarrow{BC}$

(c) $\overrightarrow{AD}$ (d) $\overrightarrow{CF}$

(e) $\overrightarrow{AC}$ (f) $\overrightarrow{BC} + \overrightarrow{DE} + \overrightarrow{FA}$

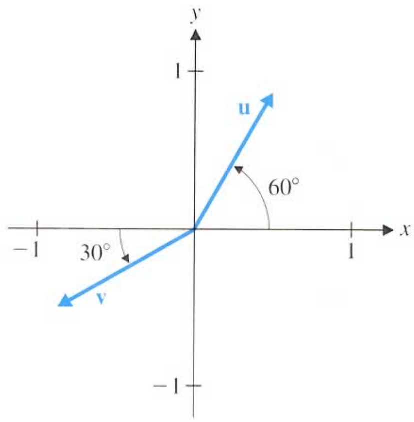

Figure 1.23

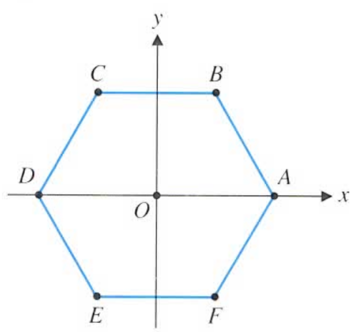

Figure 1.24

In Exercises 15 and 16, simplify the given vector expression. Indicate which properties in Theorem 1.1 you use.

15. $2(\mathbf{a} - 3\mathbf{b}) + 3(2\mathbf{b} + \mathbf{a})$

16. $-3(\mathbf{a} - \mathbf{c}) + 2(\mathbf{a} + 2\mathbf{b}) + 3(\mathbf{c} - \mathbf{b})$

In Exercises 17 and 18, solve for the vector **x** *in terms of the vectors* **a** *and* **b**.

17. $\mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a})$

18. $\mathbf{x} + 2\mathbf{a} - \mathbf{b} = 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b})$

In Exercises 19 and 20, draw the coordinate axes relative to **u** *and* **v** *and locate* **w**.

19. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$

20. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{w} = -\mathbf{u} - 2\mathbf{v}$

In Exercises 21 and 22, draw the standard coordinate axes on the same diagram as the axes relative to **u** *and* **v**. *Use these to find* **w** *as a linear combination of* **u** *and* **v**.

21. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

22. $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

23. Draw diagrams to illustrate properties (d) and (e) of Theorem 1.1.

24. Give algebraic proofs of properties (d) through (g) of Theorem 1.1.

In Exercises 25–28, **u** *and* **v** *are binary vectors. Find* $\mathbf{u} + \mathbf{v}$ *and* $\mathbf{u} \cdot \mathbf{v}$ *in each case.*

25. $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

26. $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

27. $\mathbf{u} = [1, 0, 1, 1], \mathbf{v} = [1, 1, 1, 1]$

28. $\mathbf{u} = [1, 1, 0, 1, 0], \mathbf{v} = [0, 1, 1, 1, 0]$

29. Write out the addition and multiplication tables for $\mathbb{Z}_4$.

30. Write out the addition and multiplication tables for $\mathbb{Z}_5$.

In Exercises 31–43, perform the indicated calculations.

31. $2 + 2 + 2$ in $\mathbb{Z}_3$

32. $2 \cdot 2 \cdot 2$ in $\mathbb{Z}_3$

33. $2(2 + 1 + 2)$ in $\mathbb{Z}_3$

34. $3 + 1 + 2 + 3$ in $\mathbb{Z}_4$

35. $2 \cdot 3 \cdot 2$ in $\mathbb{Z}_4$

36. $3(3 + 3 + 2)$ in $\mathbb{Z}_4$

37. $2 + 1 + 2 + 2 + 1$ in $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_5$

38. $(3 + 4)(3 + 2 + 4 + 2)$ in $\mathbb{Z}_5$

39. $8(6 + 4 + 3)$ in $\mathbb{Z}_9$

40. 2^{100} in $\mathbb{Z}_{11}$

41. $[2, 1, 2] + [2, 0, 1]$ in $\mathbb{Z}_3^3$

42. $[2, 1, 2] \cdot [2, 2, 1]$ in $\mathbb{Z}_3^3$

43. $[2, 0, 3, 2] \cdot ([3, 1, 1, 2] + [3, 3, 2, 1])$ in $\mathbb{Z}_4^4$ and in $\mathbb{Z}_5^4$

In Exercises 44–55, solve the given equation or indicate that there is no solution.

44. $x + 3 = 2$ in $\mathbb{Z}_5$

45. $x + 5 = 1$ in $\mathbb{Z}_6$

46. $2x = 1$ in $\mathbb{Z}_3$

47. $2x = 1$ in $\mathbb{Z}_4$

48. $2x = 1$ in $\mathbb{Z}_5$

49. $3x = 4$ in $\mathbb{Z}_5$

50. $3x = 4$ in $\mathbb{Z}_6$

51. $6x = 5$ in $\mathbb{Z}_8$

52. $8x = 9$ in $\mathbb{Z}_{11}$

53. $2x + 3 = 2$ in $\mathbb{Z}_5$

54. $4x + 5 = 2$ in $\mathbb{Z}_6$

55. $6x + 3 = 1$ in $\mathbb{Z}_8$

56. (a) For which values of a does $x + a = 0$ have a solution in $\mathbb{Z}_5$?
(b) For which values of a and b does $x + a = b$ have a solution in $\mathbb{Z}_6$?
(c) For which values of a, b, and m does $x + a = b$ have a solution in $\mathbb{Z}_m$?

57. (a) For which values of a does $ax = 1$ have a solution in $\mathbb{Z}_5$?
(b) For which values of a does $ax = 1$ have a solution in $\mathbb{Z}_6$?
(c) For which values of a and m does $ax = 1$ have a solution in $\mathbb{Z}_m$?