Exercises 1.2

In Exercises 1–6, find $\mathbf{u}\cdot\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$

3. $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

CAS 4. $\mathbf{u} = \begin{bmatrix} 1.5 \\ 0.4 \\ -2.1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3.0 \\ 5.2 \\ -0.6 \end{bmatrix}$

5. $\mathbf{u} = [1, \sqrt{2}, \sqrt{3}, 0], \mathbf{v} = [4, -\sqrt{2}, 0, -5]$

CAS 6. $\mathbf{u} = [1.12, -3.25, 2.07, -1.83]$,
$\mathbf{v} = [-2.29, 1.72, 4.33, -1.54]$

In Exercises 7–12, find $\|\mathbf{u}\|$ *for the given exercise, and give a unit vector in the direction of* $\mathbf{u}$.

7. Exercise 1 **8.** Exercise 2 **9.** Exercise 3

CAS **10.** Exercise 4 **11.** Exercise 5 CAS **12.** Exercise 6

In Exercises 13–16, find the distance $d(\mathbf{u}, \mathbf{v})$ *between* $\mathbf{u}$ *and* $\mathbf{v}$ *in the given exercise.*

13. Exercise 1 **14.** Exercise 2

15. Exercise 3 CAS **16.** Exercise 4

17. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbb{R}^n$, $n \geq 2$, and c is a scalar, explain why the following expressions make no sense:

(a) $\|\mathbf{u}\cdot\mathbf{v}\|$ (b) $\mathbf{u}\cdot\mathbf{v} + \mathbf{w}$
(c) $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ (d) $c\cdot(\mathbf{u} + \mathbf{w})$

In Exercises 18–23, determine whether the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *is acute, obtuse, or a right angle.*

18. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

19. $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

20. $\mathbf{u} = [5, 4, -3], \mathbf{v} = [1, -2, -1]$

CAS 21. $\mathbf{u} = [0.9, 2.1, 1.2], \mathbf{v} = [-4.5, 2.6, -0.8]$

22. $\mathbf{u} = [1, 2, 3, 4], \mathbf{v} = [-3, 1, 2, -2]$

23. $\mathbf{u} = [1, 2, 3, 4], \mathbf{v} = [5, 6, 7, 8]$

In Exercises 24–29, find the angle between $\mathbf{u}$ *and* $\mathbf{v}$ *in the given exercise.*

24. Exercise 18 **25.** Exercise 19 **26.** Exercise 20

CAS **27.** Exercise 21 CAS **28.** Exercise 22 CAS **29.** Exercise 23

30. Let $A = (-3, 2)$, $B = (1, 0)$, and $C = (4, 6)$. Prove that ΔABC is a right-angled triangle.

31. Let $A = (1, 1, -1)$, $B = (-3, 2, -2)$, and $C = (2, 2, -4)$. Prove that ΔABC is a right-angled triangle.

CAS **32.** Find the angle between a diagonal of a cube and an adjacent edge.

33. A cube has four diagonals. Show that no two of them are perpendicular.

In Exercises 34–39, find the projection of $\mathbf{v}$ *onto* $\mathbf{u}$. *Draw a sketch in Exercises 34 and 35.*

34. A parallelogram has diagonals determined by the vectors

$$\mathbf{d}_1 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \text{ and } \mathbf{d}_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Show that the parallelogram is a rhombus (all sides of equal length) and determine the side length.

35. The rectangle $ABCD$ has vertices at $A = (1, 2, 3)$, $B = (3, 6, -2)$, and $C = (0, 5, -4)$. Determine the coordinates of vertex D.

36. An airplane heading due east has a velocity of 200 miles per hour. A wind is blowing from the north at 40 miles per hour. What is the resultant velocity of the airplane?

37. A boat heads north across a river at a rate of 4 miles per hour. If the current is flowing east at a rate of 3 miles per hour, find the resultant velocity of the boat.

38. Ann is driving a motorboat across a river that is 2 km wide. The boat has a speed of 20 km/h in still water, and the current in the river is flowing at 5 km/h. Ann heads out from one bank of the river for a dock directly across from her on the opposite bank. She drives the boat in a direction perpendicular to the current.

(a) How far downstream from the dock will Ann land?
(b) How long will it take Ann to cross the river?

39. Bert can swim at a rate of 2 miles per hour in still water. The current in a river is flowing at a rate of 1 mile per hour. If Bert wants to swim across the river to a point directly opposite, at what angle to the bank of the river must he swim?

In Exercises 40–45, find the projection of **v** *onto* **u**. *Draw a sketch in Exercises 40 and 41.*

40. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

41. $\mathbf{u} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

42. $\mathbf{u} = \begin{bmatrix} 2/3 \\ -2/3 \\ -1/3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$

43. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ -1 \\ -2 \end{bmatrix}$

CAS **44.** $\mathbf{u} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2.1 \\ 1.2 \end{bmatrix}$

CAS **45.** $\mathbf{u} = \begin{bmatrix} 3.01 \\ -0.33 \\ 2.52 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1.34 \\ 4.25 \\ -1.66 \end{bmatrix}$

Figure 1.39 suggests two ways in which vectors may be used to compute the area of a triangle. The area $\mathcal{A}$ of

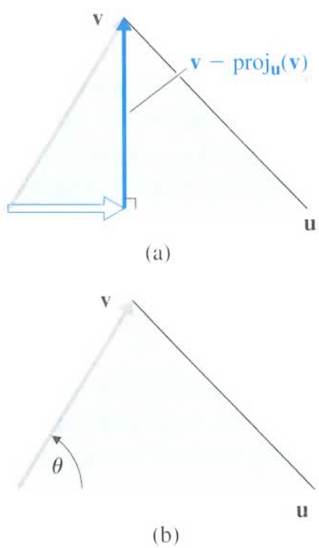

Figure 1.39

the triangle in part (a) is given by $\frac{1}{2}\|\mathbf{u}\|\|\mathbf{v} - \text{proj}_{\mathbf{u}}(\mathbf{v})\|$, and part (b) suggests the trigonometric form of the area of a triangle: $\mathcal{A} = \frac{1}{2}\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$ (We can use the identity $\sin\theta = \sqrt{1 - \cos^2\theta}$ to find $\sin\theta$.)

In Exercises 46 and 47, compute the area of the triangle with the given vertices using both methods.

46. $A = (1, -1), B = (2, 2), C = (4, 0)$

47. $A = (3, -1, 4), B = (4, -2, 6), C = (5, 0, 2)$

In Exercises 48 and 49, find all values of the scalar k for which the two vectors are orthogonal.

48. $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} k+1 \\ k-1 \end{bmatrix}$

49. $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix}$

50. Describe all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

51. Describe all vectors $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}$.

52. Under what conditions are the following true for vectors **u** and **v** in $\mathbb{R}^2$ or $\mathbb{R}^3$?

(a) $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ **(b)** $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$

53. Prove Theorem 1.2(b).

54. Prove Theorem 1.2(d).

In Exercises 55–57, prove the stated property of distance between vectors.

55. $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$ for all vectors **u** and **v**

56. $d(\mathbf{u}, \mathbf{w}) \le d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, \mathbf{w})$ for all vectors **u**, **v**, and **w**

57. $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$

58. Prove that $\mathbf{u} \cdot c\mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$ for all vectors **u** and **v** in $\mathbb{R}^n$ and all scalars c.

59. Prove that $\|\mathbf{u} - \mathbf{v}\| \ge \|\mathbf{u}\| - \|\mathbf{v}\|$ for all vectors **u** and **v** in $\mathbb{R}^n$. [*Hint:* Replace **u** by $\mathbf{u} - \mathbf{v}$ in the Triangle Inequality.]

60. Suppose we know that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$. Does it follow that $\mathbf{v} = \mathbf{w}$? If it does, give a proof that is valid in $\mathbb{R}^n$; otherwise, give a *counterexample* (that is, a *specific* set of vectors **u**, **v**, and **w** for which $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ but $\mathbf{v} \ne \mathbf{w}$).

61. Prove that $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for all vectors **u** and **v** in $\mathbb{R}^n$.

62. **(a)** Prove that $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$ for all vectors **u** and **v** in $\mathbb{R}^n$.

(b) Draw a diagram showing **u**, **v**, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ in $\mathbb{R}^2$ and use (a) to deduce a result about parallelograms.

63. Prove that $\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2$ for all vectors **u** and **v** in $\mathbb{R}^n$.

64. (a) Prove that $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\|$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
(b) Draw a diagram showing $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ in $\mathbb{R}^2$ and use (a) to deduce a result about parallelograms.

65. (a) Prove that $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are orthogonal in $\mathbb{R}^n$ if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.
(b) Draw a diagram showing $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ in $\mathbb{R}^2$ and use (a) to deduce a result about parallelograms.

66. If $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = \sqrt{3}$, and $\mathbf{u} \cdot \mathbf{v} = 1$, find $\|\mathbf{u} + \mathbf{v}\|$.

67. Show that there are no vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\|\mathbf{u}\| = 1$, $\|\mathbf{v}\| = 2$, and $\mathbf{u} \cdot \mathbf{v} = 3$.

68. (a) Prove that if $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{v} + \mathbf{w}$.
(b) Prove that if $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $s\mathbf{v} + t\mathbf{w}$ for all scalars s and t.

69. Prove that $\mathbf{u}$ is orthogonal to $\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$ for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, where $\mathbf{u} \neq \mathbf{0}$.

70. (a) Prove that $\mathrm{proj}_{\mathbf{u}}(\mathrm{proj}_{\mathbf{u}}(\mathbf{v})) = \mathrm{proj}_{\mathbf{u}}(\mathbf{v})$.
(b) Prove that $\mathrm{proj}_{\mathbf{u}}(\mathbf{v} - \mathrm{proj}_{\mathbf{u}}(\mathbf{v})) = \mathbf{0}$.
(c) Explain (a) and (b) geometrically.

71. The Cauchy-Schwarz Inequality $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|$ is equivalent to the inequality we get by squaring both sides: $(\mathbf{u} \cdot \mathbf{v})^2 \leq \|\mathbf{u}\|^2\,\|\mathbf{v}\|^2$.

(a) In $\mathbb{R}^2$, with $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, this becomes

$$(u_1v_1 + u_2v_2)^2 \leq (u_1^2 + u_2^2)(v_1^2 + v_2^2)$$

Prove this algebraically. [*Hint:* Subtract the left-hand side from the right-hand side and show that the difference must necessarily be nonnegative.]
(b) Prove the analogue of (a) in $\mathbb{R}^3$.

72. Another approach to the proof of the Cauchy-Schwarz Inequality is suggested by Figure 1.40, which shows that in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\|\mathrm{proj}_{\mathbf{u}}(\mathbf{v})\| \leq \|\mathbf{v}\|$. Show that this is equivalent to the Cauchy-Schwarz Inequality.

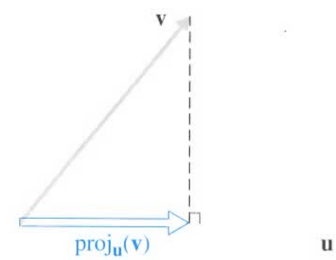

Figure 1.40

73. Use the fact that $\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = c\mathbf{u}$ for some scalar c, together with Figure 1.41, to find c and thereby derive the formula for $\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$.

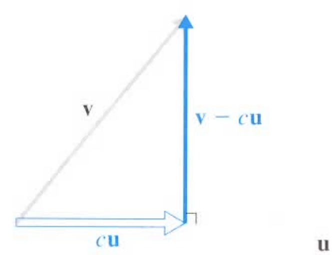

Figure 1.41

74. Using mathematical induction, prove the following generalization of the Triangle Inequality:

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_n\|$$

for all $n \geq 1$.