Answers to Selected Odd-Numbered Exercises

Appendix A

Exercises A

1. 1, 2, 3, 4 **3.** $\{1, 2\}, \{3, 4\}$

5. none **7.** $-2, -1, 0, 1, 2, 3$

9. 1, 2 **11.** none

13. $-3, -2, -1, 0, 1, 2, 3, 4, 6$

15. $\{1, 3, 5, 7, \ldots\}$ **17.** $\{\ldots, -5, -1, 3, 7, \ldots\}$

19. $\{n \in \mathbb{Z} : |n| \leq 3\}$ **21.** $\{3n - 2 : n \in \mathbb{N}\}$

23. $\{5n : n \in \mathbb{Z}\}$ **25.** $A = B = C, D = E$

27. $A \cap B = \{2, 4\}, A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$

29. $A \cap B = \varnothing, A \cup B = \{1, 2, 3, 5, 7, 10, 15, 17, 26, 31\}$

31. $\displaystyle\sum_{k=1}^{25} 2k$ **33.** $\displaystyle\sum_{k=1}^{8} 1/2^{k-1}$

35. $\displaystyle\sum_{k=1}^{11} (3k + 2)$ **37.** $\displaystyle\sum_{k=1}^{n+1} r^{k-1}$

39. $$\sum_{i=0}^{2}\sum_{j=0}^{2} |i - j| = \sum_{i=0}^{2} (|i| + |i - 1| + |i - 2|)$$
$$= (0 + 1 + 2) + (1 + 0 + 1) + (2 + 1 + 0) = 8,$$
$$\sum_{j=0}^{2}\sum_{i=0}^{2} |i - j| = \sum_{j=0}^{2} (|-j| + |1 - j| + |2 - j|)$$
$$= (0 + 1 + 2) + (1 + 0 + 1) + (2 + 1 + 0) = 8$$

41. If n is even, then $n = 2k$ for some integer k. Then $3n - 5 = 3(2k) - 5 = 2(3k - 3) + 1$, which is odd.

43. If n is even, then $n = 2k$ for some integer k and so $n^3 - n = (2k)^3 - 2k = 2(4k^3 - k)$, which is even. If n is odd, then $n = 2k + 1$ for some integer k and so $n^3 - n = (2k + 1)^3 - (2k + 1) = 8k^3 + 12k^2 + 4k = 2(4k^3 + 6k^2 + 2k)$, which is even.

45. (By contrapositive) If n is not odd, then n is even. Thus, $n = 2k$ for some integer k and so $3n + 1 = 3(2k) + 1 = 2(3k) + 1$, which is odd. Hence $3n + 1$ is not even.

47. (By contradiction) Suppose that $m + n$ is odd but it is not the case that one of m or n is even and the other is odd. Then either m and n are both even or they are both odd. In either case, $m + n$ is even, a contradiction. Conclude that one of m or n is even and the other is odd.

49. [$\Rightarrow$] (By contrapositive) Assume that it is not the case that both m and n are even. If m and n are both odd, then mn is odd; if only one of m or n is even, then $m + n$ is odd. In either case, it is not the case that mn and $m + n$ are both even.

[$\Leftarrow$] Assume that both m and n are even. Then clearly mn and $m + n$ are both even.

51. If $\sqrt{2}$ is rational, then it can be written in the form $\sqrt{2} = a/b$, where a and b are integers with no common factors. Then $2 = a^2/b^2$, so $a^2 = 2b^2$ and, hence, a^2 is even. Hence, a is even (by Exercise 49 with $m = n = a$) and so $a = 2k$ for some integer k. But then $4k^2 = 2b^2$, so $b^2 = 2k^2$ and, hence b^2 is even. Hence, b is even and a and b have a common factor of 2, a contradiction. Conclude that $\sqrt{2}$ must be irrational.

Appendix B

Exercises B

1. For $n = 1$, we have $1 = 2 \cdot 1^2 - 1$. Assume that $1 + 5 + 9 + \cdots + (4k - 3) = 2k^2 - k$. Then

$1 + 5 + \cdots + (4(k+1) - 3) = 1 + 5 + \cdots + (4k - 3) + (4k + 1) = 2k^2 - k + 4k + 1 = 2(k^2 + 2k + 1) - (k + 1) = 2(k + 1)^2 - (k + 1)$.

3. For $n = 1$, we have $1^2 = 1 = 1(1 + 1)(2 \cdot 1 + 1)/6$. Assume that $1^2 + 2^2 + 3^2 + \cdots + k^2 = k(k + 1)(2k + 1)/6$. Then $1^2 + 2^2 + \cdots + k^2 + (k + 1)^2 = k(k + 1)(2k + 1)/6 + (k + 1)^2 = (k(k + 1)(2k + 1) + 6(k + 1)^2)/6 = (k + 1)(k(2k + 1) + 6(k + 1))/6 = (k + 1)(2k^2 + 7k + 6)/6 = (k + 1)(k + 2)(2(k + 1) + 1)/6$.

5. For $n = 0$, we have $1 = 2^{0+1}-1$. Assume that $1 + 2 + 4 + 8 + \cdots + 2^k = 2^{k+1}-1$. Then $1 + 2 + 4 + \cdots + 2^k + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$.

7. For $n = 1$, we have $1 \cdot 1! = 1 = 2 - 1 = (1 + 1)! - 1$. Assume that $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$. Then $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k + 1)(k + 1)! = ((k + 1)! - 1) + (k + 1)(k + 1)! = (k + 1)!(1 + (k + 1)) - 1 = (k + 1)!(k + 2) - 1 = (k + 2)! - 1$.

9. For $n = 0$, $0^2 + 0 = 0$ is even. Assume that $k^2 + k$ is even, so that $k^2 + k = 2m$ for some integer m. Then $(k + 1)^2 + (k + 1) = k^2 + 2k + 1 + k + 1 = (k^2 + k) + 2k + 2 = 2m + 2(k + 1) = 2(m + k + 1)$, which is even.

11. For $n = 0$, $5^0-1 = 1-1 = 0$, which is divisible by 4. Assume that 5^k-1 is divisible by 4, so that $5^k-1 = 4m$ for some integer m. Then $5^{k+1}-1 = 5^{k+1}-5^k + 5^k-1 = (5-1)5^k + (5^k-1) = 4 \cdot 5^k + 4m = 4(5^k + m)$, which is divisible by 4.

13. For $n = 5$, we have $2^5 = 32 > 25 = 5^2$. Assume that $2^k > k^2$. Then $2^{k+1} = 2 \cdot 2^k > 2k^2$. But, since $k \geq 5$, $k(k-2) \geq 1$ so $k^2 \geq 2k + 1$ and hence $2k^2 \geq k^2 + 2k + 1 = (k + 1)^2$. It follows that $2^{k+1} > (k + 1)^2$.

15. For $n = 1$, we have $1 = 2 - \frac{1}{1}$. Assume that $1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{k^2} \leq 2-\frac{1}{k}$. Then $1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \left(\frac{(k+1)^2-k}{k(k+1)^2}\right)$.

Now $(k + 1)^2-k = k^2 + k + 1 \geq k^2 + k = k(k + 1)$.

Therefore, $\frac{(k+1)^2-k}{k(k+1)^2} \geq \frac{(k+1)}{k(k+1)^2} = \frac{1}{(k+1)}$. It follows that $1 + \frac{1}{4} + \cdots + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{(k+1)}$.

17. For $n = 0$, we have $(ab)^0 = 1 = 1 \cdot 1 = a^0b^0$. Assume that $(ab)^k = a^kb^k$. Then $(ab)^{k+1} = (ab)^k(ab) = a^kb^kab = a^kab^kb = a^{k+1}b^{k+1}$.

19. For $n = 1$, we have $x^1-1 = x-1$, which is certainly divisible by $x-1$. Assume that x^k-1 is divisible by $x-1$, so that $x^k-1 = (x-1)f(x)$, for some polynomial $f(x)$. Then $x^{k+1}-1 = x^{k+1}-x^k + x^k-1 = (x-1)x^k + (x-1)f(x) = (x-1)(x^k + f(x))$, which is divisible by $x-1$.

21. For $n = 0$, we have a set with no elements: the empty set $\emptyset$. The only subset of $\emptyset$ is $\emptyset$ itself, so $\emptyset$ has $1 = 2^0$ subsets. Assume that any set with k elements has 2^k subsets. Now let S be a set with 2^{k+1} elements, say $S = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$. If $A \subseteq S$, then either $x_{k+1} \in A$ or $x_{k+1} \notin A$. If $x_{k+1} \in A$, then $A = \{x_{k+1}\} \cup A'$, where A' is a subset of $\{x_1, x_2, \ldots, x_k\}$; by the induction hypothesis, there are 2^k such subsets. If $x_{k+1} \notin A$, then A is a subset of $\{x_1, x_2, \ldots, x_k\}$; by the induction hypothesis, there are 2^k such subsets. It follows that the total number of subsets of S is $2 \cdot 2^{k+1} = 2^{k+2}$, as was required to be proved.

23. *Hint*: The basis step is for $n = 3$, in which case we have a triangle and the sum of its interior angles is $180^\circ = (3-2)180^\circ$. Assuming that a convex k-gon has an interior angle sum of $(k-2)180^\circ$, consider a convex $(k + 1)$-gon P. Subdivide P into a triangle and a k-gon.

25. $n/(n + 1)$.

27. For $n = 1$, we have $1 = 2^0 \cdot 1$. Assume that, for all integers n such that $1 \leq n \leq k$, n can be factored as $n = 2^im$ for some integer $i \geq 0$ and some odd integer m. Consider $k + 1$. If $k + 1$ is odd, then $k + 1 = 2^0(k + 1)$ is the required factorization. If $k + 1$ is even, then $k + 1 = 2a$ for some integer a. Since $1 \leq k$, $k + 1 \leq 2k$ and so $a = (k + 1)/2 \leq k$. By the induction hypothesis, $a = 2^im$ for some integer $i \geq 0$ and some odd integer m. Then $k + 1 = 2a = 2^{i+1}m$ is the desired factorization.

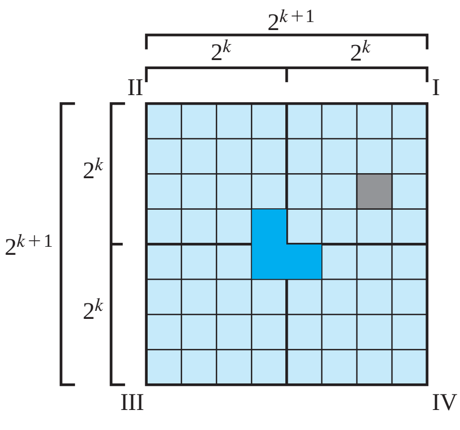
29. For $n = 8$, we have $8 = 3 \cdot 1 + 5 \cdot 1$. Assume that, for all integers n such that $8 \leq n \leq k$, n can be written as $n = 3a + 5b$ for some nonnegative integers a and b. Consider $k + 1$. Since $9 = 3 \cdot 3 + 5 \cdot 0$ and $10 = 3 \cdot 0 + 5 \cdot 2$, we may assume that $k + 1 \geq 11$. Hence $8 \leq (k + 1)-3 = k-2 \leq k$ and so $k-2 = 3a + 5b$ for some nonnegative integers a and b, by the induction hypothesis. Then $k + 1 = (k-2) + 3 = (3a + 5b) + 3 = 3(a + 1) + 5b$, as required.

31. For $n = 0$, we have $f_0 = 0 = 1-1 = f_2-1$. Assume that $\sum_{i=0}^{k} f_i = f_{k+2}-1$. Then $\sum_{i=0}^{k+1} f_i = \left(\sum_{i=0}^{k} f_i\right) + f_{k+1}$ $= f_{(k+2)}-1 + f_{k+1} = (f_{k+1} + f_{k+2}) - 1 = f_{k+3} - 1$ $= f_{(k+1)+2} - 1$.

33. For $n = 0$, we have $f_0^2 = 0^2 = 0 = 0 \cdot 1 = f_0 f_1$. Assume that $\sum_{i=0}^{k} f_i^2 = f_k f_{k+1}$. Then $\sum_{i=0}^{k+1} f_i^2 = \left(\sum_{i=0}^{k} f_i^2\right) + f_{k+1}^2$ $= f_k f_{k+1} + f_{k+1}^2 = f_{k+1}(f_k + f_{k+1}) = f_{k+1} f_{k+2}$.

35. For $n = 0$, we have $f_{m-1}f_0 + f_m f_1 = f_{m-1} \cdot 0 + f_m \cdot 1$ $= f_m = f_{m+0}$. Assume that $f_{m-1}f_n + f_m f_{n+1} = f_{m+n}$ for all $0 \le n \le k$. Then $f_{m-1}f_{k+1} + f_m f_{k+2} =$ $f_{m-1}(f_{k-1} + f_k) + f_m(f_k + f_{k+1}) = (f_{m-1}f_{k-1} + f_m f_k)$ $+ (f_{m-1}f_k + f_m f_{k+1}) = f_{m+k-1} + f_{m+k} = f_{m+k+1}$.

37. For $n = 1$, a 2×2 board with a square removed is just a single L-tile. Assume that a $2^k \times 2^k$ with a square removed can be tiled with L-tiles. Consider a $2^{k+1} \times 2^{k+1}$ board with a square removed. Subdivide the board into four $2^k \times 2^k$ quadrants. One of the quadrants contains the missing square, so it can be tiled with L-tiles, by the induction hypothesis. Now place a single L-tile at the center of the board so that it covers one square in each of the remaining three quadrants. By the induction hypothesis, the remaining squares in each quadrant can be tiled with L-tiles, and we are done.



39. For $n = 1$, clearly a single disk can be transferred to a different peg in $1 = 2^1-1$ move. Assume that a tower of k disks can be transferred to a different peg in 2^k-1 moves. Consider a tower of $k + 1$ disks on peg A, say. In order to move it to peg B, we need to move the largest disk to peg B. To do this, we first must transfer the top k disks to peg C; this takes 2^k-1 moves, by the induction hypothesis. Now move the largest disk to peg B (1 move) and then transfer the tower of k disks from peg C to peg B (2^k-1 moves). The total number of moves is thus $2(2^k - 1) + 1 = 2^{k+1} - 1$, as required.

41. The basis step is not true.

Appendix C

Exercises C

1. $8-4i$ **3.** $13 + 11i$

5. $7-4i$ **7.** $\frac{1}{2}-\frac{1}{2}i$

9. $\frac{1}{10}-\frac{13}{10}i$ **11.** $-i$

13. $10i$ **15.** 5 **17.** 3

19. $2\sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$

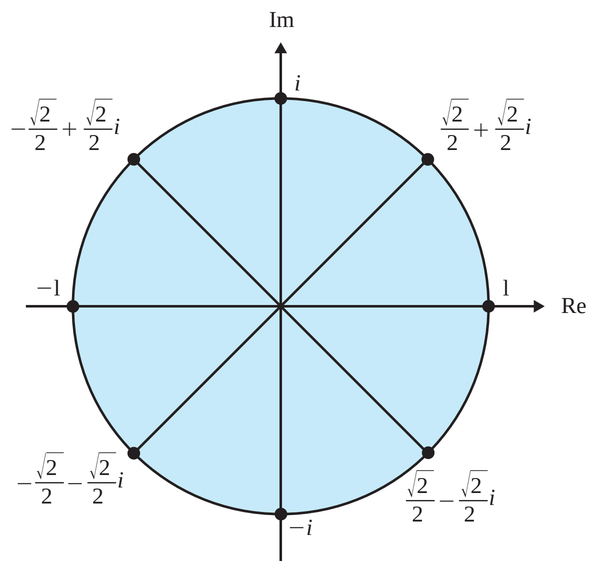
21. $2(\cos(\pi/6) + i\sin(\pi/6))$

23. $zw = 2\sqrt{2}(\cos(11\pi/12) + i\sin(11\pi/12))$, $z/w = (2/\sqrt{2})\,(\cos(7\pi/12) + i\sin(7\pi/12))$, $1/z = (\sqrt{2}/2)$ $(\cos(3\pi/4) - i\sin(3\pi/4))$

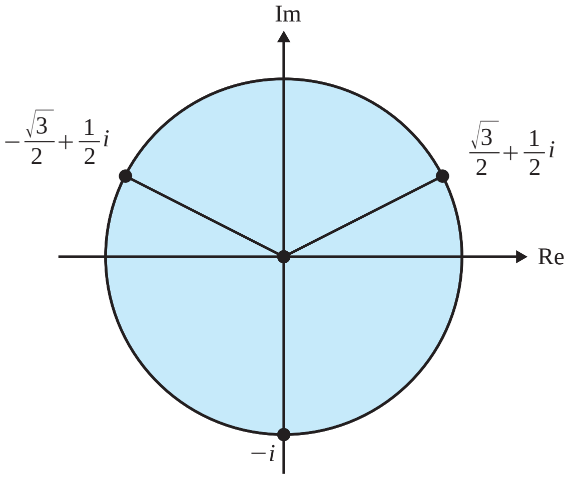
25. $zw = 8\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$, $z/w = 2/\sqrt{2}\,(\cos(\pi/4) + i\sin(\pi/4))$, $1/z = (\sqrt{2}/8)$ $(\cos(\pi/4) - i\sin(\pi/4))$

27. 16 **29.** $16-16\sqrt{3}\,i$

31. $\pm 1, \pm i, \pm(\sqrt{2}/2) \pm (\sqrt{2}/2)i$



33. $\pm\sqrt{3}/2 + i/2, -i$



35. $-i$ **37.** e

39. **(a)** $\overline{\overline{a + bi}} = \overline{a-bi} = a + bi$

(c) $\overline{(a + bi)(c + di)} = \overline{(ac-bd) + (ad + bc)i}$
$= (ac-bd)-(ad + bc)i = (a - bi)(c - di)$
$= \overline{(a + bi)}\,\overline{(c + di)}$

(e) Let $z = a + bi$. If z is real, then $b = 0$ and hence $\bar{z} = \bar{a} = a = z$. Conversely, if $z = \bar{z}$, then $a + bi = a-bi$. Thus, $2bi = 0$ and so $b = 0$. Hence $z = a$ is real.

41. **(a)** From $\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + i(2\cos\theta\sin\theta)$ we see that $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\cos\theta\sin\theta$.

(c) From $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$ $= (\cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$, we see that $\cos 4\theta = \cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta$ and $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

Appendix D

Exercises D

1. polynomial **3.** polynomial

5. polynomial for $x \neq 0$ **7.** not a polynomial

9. not a polynomial **11.** not a polynomial

13. polynomial

15. $f(x) + g(x) = 3x^2 + x - 1, f(x) - g(x) = -3x^2 + x - 3, f(x)g(x) = 3x^3 - 6x^2 + x - 2$

17. $f(x) + g(x) = x^3 + x^2 + 2x, f(x) - g(x) = -x^3 - x^2 - 2, f(x)g(x) = x^4 - 1$

19. $f(x) + g(x) = x^4 + (1 + \sqrt{2})x^2 - \sqrt{2}x + 2$, $f(x) - g(x) = x^4 + (\sqrt{2} - 1)x^2 + \sqrt{2}x$, $f(x)g(x) = x^6 - \sqrt{2}x^5 + (1 + \sqrt{2})x^4 - 2x^3 + (1 + \sqrt{2})x^2 - \sqrt{2}x + 1$

21. $x^2 = (x + 1)(x-1) + 1$

23. $2x^3 - x^2 = (x - 2)(2x^2 + 3x + 6) + 12$

25. $x^4 + x^3 - 3x^2 - 2x + 2 = (x^2 + x - 1)(x^2 - 2)$

27. $2/5, 2$ **29.** no rational roots

31. $-1/2 \pm \sqrt{5}/2$ **33.** $-1, \pm\sqrt{2}$

35. $-1, 2, 4 \pm i$

37. There is one sign change, so p has at most one positive zero. But $p(0) = -1$ and $p(1) = 1$, so there is a zero in the interval $(0, 1)$. Therefore p has exactly one positive zero.

38. There are no sign changes, so p has no positive zeros. Since $p(-x) = -2x^3 + 3x^2 + 4$ has one sign change, p has at most one negative zero. We find that $p(-2) = 0$ and so, since $p(0) \neq 0$, p has exactly one real zero. Since p has degree 3, p has three zeros altogether. Hence, p has exactly two complex (nonreal) zeros.

41. There is one sign change so p has at most one positive zero. Since $p(0) = -1$ and $p(1) = 2$, there is a zero in the interval $(0, 1)$. Since $p(-x) = x^4 + 5x^2 + 3x-1$ also has one sign change, p has at most one negative zero. From $p(-1) = 8$ and $p(0) = -1$, we see that there is a zero in the interval $(-1, 0)$. Since $p(0) \neq 0$, p has exactly two real zeros and so, because p has degree 4, it must have two complex (nonreal) zeros as well.

43. If $p(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n$, then $p^*(x) = a_n + a_{n-1}x + \cdots + a_1x^{n-1} + a_0x^n$.

45. First note that x^{-1} makes sense since $x = 0$ is not a solution of $p(x) = 0$, by Exercise 44(a). Since p is palindromic of degree $2n$, we have $p(x) = a_0x^{2n} + a_1x^{2n-1} + \cdots + a_nx^n + \cdots + a_1x + a_0$, so, multiplying by x^{-n}, $p(x) = 0$ can be rewritten as $a_0x^n + a_1x^{n-1} + \cdots + a_n + \cdots + a_1x^{-(n-1)} + a_0x^{-n} = 0$, or $a_0(x^n + x^{-n}) + a_1(x^{n-1} + x^{-(n-1)}) + \cdots + a_n = 0$. It is now enough to prove that $x^n + x^{-n}$ is a polynomial of degree n in $t = x + x^{-1}$ for all $n \geq 1$. For $n = 1$, it is clear. Assume that $x^n + x^{-n}$ is a polynomial of degree n in t, for all $1 \leq n \leq k$. Then $x^k + x^{-k} = f(t)$, where f has degree k, and $x^{k-1} + x^{-(k-1)} = g(t)$, where g has degree $k-1$. Therefore, $x^{k+1} + x^{-(k+1)} = (x + x^{-1})(x^k + x^{-k}) - (x^{k-1} + x^{-(k-1)}) = tf(t)-g(t)$, which is a polynomial in t of degree $k + 1$.

47. **(a)** From $\alpha = e^{2\pi i/5} = \cos(2\pi/5) + i\sin(2\pi/5)$ and $\alpha^{-1} = e^{-2\pi i/5} = \cos(-2\pi/5) + i\sin(-2\pi/5) = \cos(2\pi/5) - i\sin(2\pi/5)$, we obtain $\alpha + \alpha^{-1} = 2\cos(2\pi/5)$.

(c) $(\sqrt{5} - 1)/4$