Exercise for Section 4.6. Consider the Markov process with stochastic matrix

$$Q = \begin{bmatrix} 1/3 & 1/6 \\ 2/3 & 5/6 \end{bmatrix}.$$

(a) Find the equilibrium state $\mathbf{y}$ for a total initial population of 10.
(b) Show that in the long run any initial state

$$\mathbf{x}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$$

with $a + b = 10$ tends to the equilibrium state $\mathbf{y}$ found in (a).

Solution. (a) The equilibrium states are the eigenvectors of Q to the eigenvalue 1. We solve the linear system

$$(Q - I)\mathbf{y} = \mathbf{0}$$

and find

$$\mathbf{y} = t \begin{bmatrix} 1/4 \\ 1 \end{bmatrix}.$$

The condition $t(1/4 + 1) = 10$ gives $t = 8$, hence

$$\mathbf{y} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$

(Check that this is an equilibrium state!)
(b) We want to write $\mathbf{x}_0$ as a linear combination of eigenvectors. We already know that $\mathbf{y}$ is a basis for the eigenspace to the eigenvalue $\lambda = 1$.

To find the other eigenvalue, we compute the characteristic polynomial

$$p_Q(\lambda) = \det(Q - I_2) = \lambda^2 - \frac{7}{6}\lambda + \frac{1}{6} = (\lambda - 1)\big(\lambda - \frac{1}{6}\big).$$

Hence the other eigenvalue is $\lambda = 1/6$. A basis for the corresponding eigenspace is given by

$$\mathbf{z} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(Check!)

We now want to write

$$\mathbf{x}_0 = c\mathbf{y} + d\mathbf{z}.$$

We solve this linear system in c, d and find $c = 1$, $d = b - 8 = 2 - a$. (Check!) As a result,

$$\mathbf{x}_0 = \mathbf{y} + (2 - a)\mathbf{z}.$$

Using that $\mathbf{y}$ and $\mathbf{z}$ are eigenvectors of Q, we can compute the whole Markov chain,

$$\mathbf{x}_k = Q^k\mathbf{x}_0 = Q^k\mathbf{y} + (2 - a)Q^k\mathbf{z} = \mathbf{y} + (2 - a)(1/6)^k\mathbf{z}.$$

Now $(1/6)^k$ tends to 0 as $k \to \infty$. Hence the process tends to $\mathbf{y}$, as claimed.

Alternate solution for (b): If $\mathbf{x}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ is an initial state with $a + b = 10$, then $\frac{1}{10}\mathbf{x}_0$ is an initial probability vector. So by Theorem 4.34, since Q has positive entries, we know that

$$Q^k \frac{1}{10}\mathbf{x}_0 \quad \text{tends to} \quad \frac{1}{10}\mathbf{y},$$

since $\frac{1}{10}\mathbf{y}$ is the steady-state probability vector. Therefore

$$\frac{1}{10}Q^k \mathbf{x}_0 \quad \text{tends to} \quad \frac{1}{10}\mathbf{y},$$

and so

$$Q^k \mathbf{x}_0 \quad \text{tends to} \quad \mathbf{y}.$$