Math 1600A Lecture 11, Section 002, 2 Oct 2013

Announcements:

Read Section 2.4 for next class. Work through recommended homework questions.

Midterm 1 is tomorrow 7-8:30pm. It covers until the end of Section 2.2, except for linear systems over $\mathbb{Z}_m$. A **practice exam** is available from the course home page. Last name A-Q must write in **NS1**, R-Z in **NS7**. See the missed exam section of the course web page for policies, including for illness.

Tutorials: No quiz, focused on review. Take advantage of them! No quizzes next week either.

Office hour: today, 12:30-1:30, MC103B.

Help Centers: Monday-Friday 2:30-6:30 in MC 106.

These **lecture notes** now available in pdf format as well, a day or two after each lecture. Be sure to let me know of technical problems.

Partial review of Lecture 10:

Linear combinations

Recall: A vector $\vec{v}$ is a **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there exist scalars $c_1, c_2, \ldots, c_k$ (called coefficients) such that

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{v}.$$

Example: Is $\begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$?

That is, can we find scalars x and y such that

$$x\begin{bmatrix}4\\5\\6\end{bmatrix}+y\begin{bmatrix}2\\1\\3\end{bmatrix}=\begin{bmatrix}4\\8\\6\end{bmatrix}?$$

Expanding this into components, this becomes a linear system

$$\begin{aligned}4x+2y&=4\\5x+\ \ y&=8\\6x+3y&=6\end{aligned}\qquad\text{with augmented matrix}\qquad\left[\begin{array}{cc|c}4&2&4\\5&1&8\\6&3&6\end{array}\right]$$

and we **already know** how to determine whether this system is consistent: use **row reduction**!

Theorem 2.4: A system with augmented matrix $[A \mid \vec{b}\,]$ is consistent if and only if $\vec{b}$ is a linear combination of the columns of A.

This gives a **different** geometrical way to understand the solutions to a system.

Spanning Sets of Vectors

Definition: If $S=\{\vec{v}_1,\ldots,\vec{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of *all* linear combinations of $\vec{v}_1,\ldots,\vec{v}_k$ is called the **span** of $\vec{v}_1,\ldots,\vec{v}_k$ and is denoted $\mathrm{span}(\vec{v}_1,\ldots,\vec{v}_k)$ or $\mathrm{span}(S)$.
If $\mathrm{span}(S)=\mathbb{R}^n$, then S is called a **spanning set** for $\mathbb{R}^n$.

Example: $\mathrm{span}(\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n)=\mathbb{R}^n$.

Example: The span of $\vec{u}=\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}4\\5\\6\end{bmatrix}$ consists of every vector $\vec{x}$ that can be written as

$$\vec{x}=s\vec{u}+t\vec{v}$$

for some scalars s and t. Since $\vec{u}$ and $\vec{v}$ are not parallel, this is the plane through the origin in $\mathbb{R}^3$ with direction vectors $\vec{u}$ and $\vec{v}$.

New material: Section 2.3: Spanning Sets and Linear Independence

Question: What is $\text{span}(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$? What is $\text{span}(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix})$?

Question: We saw that $\text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \mathbb{R}^2$. What is $\text{span}(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix})$?

Question: What vector is always in $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$?

Linear Dependence and Independence

Suppose that we have vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ such that $2\vec{u} + 3\vec{v} - 2\vec{w} = \vec{0}$. This can be solved for any of the vectors in terms of the others, e.g. $\vec{u} = -\frac{3}{2}\,\vec{v} + \vec{w}$. We say that these vectors are **linearly dependent**.

Definition: A set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ is **linearly dependent** if there are scalars $c_1, \ldots, c_k$, <u>at least one of which is nonzero</u>, such that

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}.$$

Since at least one of the scalars is non-zero, the corresponding vector can be expressed as a linear combination of the others.

Example: $\begin{bmatrix} -2 \\ 4 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so the vectors $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ are linearly dependent.

Note that either of the first two can be expressed as a linear combination of the other, but the third one is not a linear combination of the first two.

Example: Are the vectors $\vec{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ linearly dependent?

Solution: If $c\,\vec{e}_1 + d\,\vec{e}_2 = \vec{0}$ and $c \neq 0$, then $\vec{e}_1 = -\frac{d}{c}\,\vec{e}_2$, which is not possible.

Similarly, if $d \neq 0$, then $\vec{e}_2$ is a multiple of $\vec{e}_1$. So the only way to have $c\,\vec{e}_1 + d\,\vec{e}_2 = \vec{0}$ is with $c = d = 0$.

Theorem 2.5: The vectors $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent if and only if at least one of them can be expressed as a linear combination of the others.

Proof: We've seen one direction. For the other, if $\vec{v}_k = c_1\vec{v}_1 + \cdots c_{k-1}\vec{v}_{k-1}$, then $c_1\vec{v}_1 + \cdots c_{k-1}\vec{v}_{k-1} - \vec{v}_k = \vec{0}$, so the vectors are linearly dependent. The same argument works if it is a different vector that can be expressed in terms of the others.

Example: What about the vectors $\vec{e}_1$, $\vec{e}_2$ and $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$?

Solution: They are linearly dependent, since

$$0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Fact: Any set of vectors containing the zero vector is linearly dependent.

Definition: A set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ is **linearly <u>in</u>dependent** if it is not linearly dependent.

Another way to say this is that the system

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}.$$

has only the trivial solution $c_1 = \cdots = c_k = 0$.

This is something we know how to figure out! Use **row reduction**!

Example: Are the vectors $\vec{u} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 6 \\ -4 \\ -2 \end{bmatrix}$ linearly independent?

That is, does the system

$$c_1\begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 6 \\ -4 \\ -2 \end{bmatrix} = \vec{0}$$

have a non-trivial solution?

The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 2 & 6 & 0 \\ 3 & 1 & -4 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right] \quad \text{which row reduces to} \quad \left[\begin{array}{ccc|} -1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right.$$

So what's the answer? There are 3 variables and 2 leading variables (the rank is 2), so there is one free variable, which means there are non-trivial solutions. Therefore, the vectors are linearly **dependent**.

Example: Are the vectors $\vec{u} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$ linearly independent?

That is, does the system

$$c_1 \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix} = \vec{0}$$

have a non-trivial solution?

The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 2 & 6 & 0 \\ 3 & 1 & -4 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right] \quad \text{which row reduces to} \quad \left[\begin{array}{ccc|} -1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right.$$

So what's the answer? There are 3 variables and 3 leading variables (the rank is 3), so there are no free variables, which means there is only the trivial solution. Therefore, the vectors are linearly **independent**.

Example 2.24: Are the standard unit vectors $\vec{e}_1, \ldots, \vec{e}_n$ in $\mathbb{R}^n$ linearly independent?

Solution: The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{array}\right]$$

with n rows and n variables. The rank is n, so there is only the trivial solution. So the standard unit vectors are linearly **independent**.

Note: You can sometimes see by inspection that some vectors are linearly dependent, e.g. if they contain the zero vector, or if one is a scalar multiple of another. Here's one other situation:

Theorem 2.8: If $m > n$, then any set of m vectors in $\mathbb{R}^n$ is linearly dependent.

Proof: The system is a homogeneous system with m variables and n equations. By Theorem 2.3, a homogeneous system with more variables than equations always has a non-trivial solution.

On whiteboard: An example like Example 2.25 in the text, and a discussion of Theorem 2.7.