Math 1600A Lecture 13, Section 002, 7 Oct 2013

Announcements:

Read Sections 3.1 and 3.2 for next class. Work through recommended homework questions.

Tutorials: No quizzes this week, focused on review.

Help Centers: Monday-Friday 2:30-6:30 in MC 106.

These **lecture notes** now available in pdf format as well, a day or two after each lecture. Be sure to let me know of technical problems.

Partial review of Lecture 12:

We covered network analysis and electrical networks in Section 2.4. It turns out that the other section did not cover electrical networks, so that part of last lecture will not be on any quizzes or exams.

We also aren't covering Section 2.5.

New material: Section 3.1: Matrix Operations

(Lots of definitions, but no tricky concepts.)

Definition: A **matrix** is a rectangular array of numbers called the **entries**. The entries are usually real (from $\mathbb{R}$), but may also be complex (from $\mathbb{C}$) or be from $\mathbb{Z}_m$.

Examples:

$$A = \begin{bmatrix} 1 & -3/2 & \pi \\ \sqrt{2} & 2.3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$2 \times 3 \qquad\qquad\quad 2 \times 2 \qquad\qquad 1 \times 4 \qquad\qquad 4 \times 1$$

$$\text{square} \qquad \begin{array}{c}\textbf{row matrix}\\\textbf{or row vector}\end{array} \qquad \begin{array}{c}\textbf{column matrix}\\\textbf{or column vector}\end{array}$$

The entry in the ith row and jth column of A is usually written a_{ij} or sometimes A_{ij}. For example,

$$A_{11} = 1, \quad A_{23} = 0, \quad A_{32} \text{ doesn't make sense.}$$

Definition: An $m \times n$ matrix A is **square** if $m = n$. The **diagonal entries** are $a_{11}, a_{22}, \ldots$. If A is square and the <u>non</u>diagonal entries are all zero, then A is called a **diagonal matrix**.

$$\begin{bmatrix} 1 & -3/2 & \pi \\ \sqrt{2} & 2.3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

not square or diagonal   square   diagonal   diagonal   scalar

Definition: A diagonal matrix with all diagonal entries equal is called a **scalar matrix**. A scalar matrix with diagonal entries all equal to 1 is an **identity matrix**.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

identity matrix   scalar

Note: Identity $\implies$ scalar $\implies$ diagonal $\implies$ square.

Now we're going to mimick a lot of what we did when we first introduced vectors.

Definition: Two matrices are **equal** if they have the same size and their corresponding entries are equal.

$$\begin{bmatrix} 1 & -3/2 \\ \sqrt{2} & 0 \end{bmatrix} \qquad \begin{bmatrix} \cos 0 & -1.5 \\ \sqrt{2} & \sin 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

The first two above are equal, but no other two are equal. We distinguish row matrices from column matrices!

Matrix addition and multiplication

Our first two operations are just like for vectors:

Definition: If A and B are <u>both</u> $m \times n$ matrices, then their **sum** $A + B$ is the $m \times n$ matrix obtained by adding the corresponding entries of A and B.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ \pi & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ 4 + \pi & 5 & 0 \end{bmatrix}$$

Using the notation $A = [a_{ij}]$ and $B = [b_{ij}]$, we write

$$A + B = [a_{ij} + b_{ij}] \qquad \text{or} \qquad (A + B)_{ij} = a_{ij} + b_{ij}.$$

Definition: If A is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** cA is the $m \times n$ matrix obtained by multiplying each entry by c.

$$3 \begin{bmatrix} 0 & -1 & 2 \\ \pi & 0 & -6 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 6 \\ 3\pi & 0 & -18 \end{bmatrix}$$

We write $cA = [c\, a_{ij}]$ or $(cA)_{ij} = c\, a_{ij}$.

Definition: As expected, $-A$ means $(-1)A$ and $A - B$ means $A + (-B)$.
The $m \times n$ **zero matrix** has all entries 0 and is denoted O or $O_{m \times n}$. Of course, $A + O = A$.
So we have the real number 0, the zero vector $\vec{0}$ (or $\mathbf{0}$ in the text) and the zero matrix O.

Matrix multiplication

This is *unlike* anything we have seen for vectors.

Definition: If A is $m \times n$ and B is $n \times r$, then the **product** $C = AB$ is the $m \times r$ matrix whose i, j entry is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

This is the dot product of the ith row of A with the jth column of B.

Note that for this to make sense, the number of columns of A must equal the number of rows of B.

$$\begin{array}{ccc} A & B & = & AB \\ m \times n & n \times r & & m \times r \end{array}$$

This may seem very strange, but it turns out to be useful. We will never use componentwise multiplication, as it is not generally useful.

Examples on whiteboard: 2×3 times 3×4, 1×3 times 3×1, 3×1 times 1×3.

One motivation for this definition of matrix multiplication is that it comes up in linear systems.

Example 3.8: Consider the system

$$\begin{aligned} 4x + 2y &= 4 \\ 5x + \ \ y &= 8 \\ 6x + 3y &= 6 \end{aligned}$$

The left-hand sides are in fact a matrix product:

$$\begin{bmatrix} 4 & 2 \\ 5 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Every linear system can be written as $A\vec{x} = \vec{b}$.

Question: If A is an $m \times n$ matrix and $\vec{e}_1$ is the first standard unit vector in $\mathbb{R}^n$, what is $A\vec{e}_1$?

The answer is an $m \times 1$ column matrix, whose ith entry is the dot product of the ith row of A with the vector $\vec{e}_1$. But $[a_{i1}, a_{i2}, \ldots, a_{in}] \cdot [1, 0, \ldots, 0] = a_{i1}$, the first entry. So this just "picks out" the first column of A.

More generally, we have:

Theorem 3.1: If A is $m \times n$, $\vec{e}_i$ is the ith $1 \times m$ standard row vector and $\vec{e}_j$ is the jth $n \times 1$ standard column vector, then

$$\vec{e}_i A = \text{the } i\text{th row of } A \quad \text{and} \quad A\vec{e}_j = \text{the } j\text{th column of } A.$$

Powers

In general, $A^2 = AA$ doesn't make sense. But if A is $n \times n$ (square), then $A^2 = AA$ does make sense. A^2 is $n \times n$ as well, and so it also makes sense to define the **power**

$$A^k = AA \cdots A \quad \text{with } k \text{ factors.}$$

We write $A^1 = A$ and $A^0 = I_n$.

We will see later that $(AB)C = A(BC)$, so the expression for A^k is unambiguous. And it follows that

$$A^r A^s = A^{r+s} \qquad \text{and} \qquad (A^r)^s = A^{rs}$$

for all nonnegative integers r and s.

Example on whiteboard: Powers of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

.