# Math 1600A Lecture 19, Section 2, 23 Oct 2013

## Announcements:

Continue **reading** Section 3.5, start Section 3.6. Work through recommended homework questions.

**Tutorials:** Quiz 4 this week covers Sections 3.2, 3.3 and the beginning of Section 3.5 (up to and including Example 3.41).

**Office hour:** today, 12:30-1:30, MC103B.
**Help Centers:** Monday-Friday 2:30-6:30 in MC 106.

## Partial review of Lecture 18:

### Subspaces

**Definition:** A **subspace** of $\mathbb{R}^n$ is any collection $S$ of vectors in $\mathbb{R}^n$ such that:
1. The zero vector $\vec{0}$ is in $S$.
2. $S$ is **closed under addition**: If $\vec{u}$ and $\vec{v}$ are in $S$, then $\vec{u} + \vec{v}$ is in $S$.
3. $S$ is **closed under scalar multiplication**: If $\vec{u}$ is in $S$ and $c$ is any scalar, then $c\,\vec{u}$ is in $S$.

Conditions (2) and (3) together are the same as saying that $S$ is **closed under linear combinations**.

**Example:** $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. Also, $S = \{\vec{0}\}$ is a subspace of $\mathbb{R}^n$.

A line or plane through the origin in $\mathbb{R}^3$ is a subspace. Applet.

On the other hand, a plane **not** through the origin is not a subspace. It of course fails (1), but the other conditions fail as well, as shown in the applet.

**Theorem 3.19:** Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. Then $\mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$.

### Subspaces associated with matrices

**Theorem 3.21:** Let $A$ be an $m \times n$ matrix and let $N$ be the set of solutions of the homogeneous system $A\,\vec{x} = \vec{0}$. Then $N$ is a subspace of $\mathbb{R}^n$.

**Aside:** At this point, the book states **Theorem 3.22**, which says that every linear system has no solution, one solution or infinitely many solutions, and gives a proof of this. We already know this is true, using Theorem 2.2 from Section 2.2 (see Lecture 9). The proof given here is in a sense better, since it doesn't rely on knowing anything about row echelon form, but I won't use class time to cover it.

Spans and null spaces are the *two main* sources of subspaces.

**Definition:** Let $A$ be an $m \times n$ matrix.

1. The **row space** of $A$ is the subspace $\mathrm{row}(A)$ of $\mathbb{R}^n$ spanned by the rows of $A$.
2. The **column space** of $A$ is the subspace $\mathrm{col}(A)$ of $\mathbb{R}^m$ spanned by the columns of $A$.
3. The **null space** of $A$ is the subspace $\mathrm{null}(A)$ of $\mathbb{R}^n$ consisting of the solutions to the system $A\vec{x} = \vec{0}$.

**Example:** The column space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\mathrm{span}(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix})$, which we saw is all of $\mathbb{R}^2$. We also saw that the row space of $A$ is $\mathbb{R}^2$.

**Example:** The column space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is the span of the two columns, which is a subspace of $\mathbb{R}^3$. Since the columns are linearly independent, this is a plane through the origin in $\mathbb{R}^3$.

# New material

We will learn methods to describe the three subspaces associated to a matrix $A$. But how do we want to "describe" a subspace? That's our next topic:

## Basis

We know that to describe a plane $\mathcal{P}$ through the origin, we can give two direction vectors $\vec{u}$ and $\vec{v}$ which are linearly independent. Then $\mathcal{P} = \mathrm{span}(\vec{u}, \vec{v})$. We know that two vectors is always enough, and one vector will not work.

**Definition:** A **basis** for a subspace $S$ of $\mathbb{R}^n$ is a set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ such that:
1. $S = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k)$, and
2. $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent.

Condition (2) ensures that none of the vectors is redundant, so we aren't being wasteful. Giving a basis for a subspace is a good way to "describe" it.

**Example 3.42:** The standard unit vectors $\vec{e}_1, \ldots, \vec{e}_n$ in $\mathbb{R}^n$ are linearly independent and span $\mathbb{R}^n$, so they form a basis of $\mathbb{R}^n$ called the **standard basis**.

**Example:** We saw above that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ span $\mathbb{R}^2$. They are also linearly independent, so they are a basis for $\mathbb{R}^2$.

Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are another basis for $\mathbb{R}^2$. A subspace will in general have many bases, but we'll see soon that they all have the same number of vectors! (Grammar: one basis, two bases.)

**Example:** Let $\mathcal{P}$ be the plane through the origin with direction vectors $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$. Then $\mathcal{P}$ is a subspace of $\mathbb{R}^3$ and these two vectors are a basis for $\mathcal{P}$.

**Example:** Find a basis for $S = \text{span}(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix})$.

**Solution:**
You can see by inspection that these vectors aren't linearly independent: the third is the sum of the first two. So $S = \text{span}(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix})$. These two vectors are linearly independent, so they form a basis for $S$.

In more complicated situations, there are two ways to find a basis of the span of a set of vectors. The first way uses the following result:

**Theorem 3.20:** Let $A$ and $B$ be row equivalent matrices. Then $\text{row}(A) = \text{row}(B)$.

**Proof:** Suppose $B$ is obtained from $A$ by performing elementary row operations. Each of these operations expresses the new row as a linear combination of the previous rows. So every row of $B$ is a linear combination of the rows of $A$. So $\text{row}(B) \subseteq \text{row}(A)$.

On the other hand, each row operation is reversible, so reversing the above argument gives that $\mathrm{row}(A) \subseteq \mathrm{row}(B)$. Therefore, $\mathrm{row}(A) = \mathrm{row}(B)$. $\square$

This will be useful, because it is easy to understand the row space of a matrix in row echelon form.

**Example:** Let's redo the above example. Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ -2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

whose rows are the given vectors. So $S = \mathrm{row}(A)$.

Row reduction produces the following matrix

$$B = \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 7/3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in reduced row echelon form. By Theorem 3.20, $S = \mathrm{row}(B)$. But the first two rows clearly give a basis for $\mathrm{row}(B)$, so another solution to the question is $\begin{bmatrix} 1 \\ 0 \\ 2/3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 7/3 \end{bmatrix}$.

**Theorem:** If $A$ is a matrix in row echelon form, then the non-zero rows of $A$ form a basis for $\mathrm{row}(A)$.

**Example:** Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \\ \vec{a}_4 \end{bmatrix}$$

$\mathrm{row}(A)$ is the span of the non-zero rows, since zero rows don't contribute. So we just need to see that the non-zero rows are linearly independent. If we had $c_1\,\vec{a}_1 + c_2\,\vec{a}_2 + c_3\,\vec{a}_3 = \vec{0}$, then $c_1 = 0$, by looking at the first component. So $5c_2 = 0$, by looking at the second component. And so $8c_3 = 0$, by looking at the fourth component. So $c_1 = c_2 = c_3 = 0$.

The same argument works in general, by looking at the pivot columns, and this proves the Theorem.

This gives rise to the **row method** for finding a basis for a subspace $S$ spanned by some vectors $\vec{v}_1, \ldots, \vec{v}_k$:

1. Form the matrix $A$ whose rows are $\vec{v}_1, \ldots, \vec{v}_k$, so $S = \mathrm{row}(A)$.
2. Reduce $A$ to row echelon form $B$.
3. The non-zero rows of $B$ will be a basis of $S = \mathrm{row}(A) = \mathrm{row}(B)$.

Notice that the vectors you get are usually different from the vectors you started with. Given $S = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k)$, one can always find a basis for $S$ which just omits some of the given vectors. We'll explain this next.

Suppose we form a matrix $A$ whose <u>columns</u> are $\vec{v}_1, \ldots, \vec{v}_k$. A non-zero solution to the system $A\vec{x} = \vec{0}$ is exactly a dependency relationship between the given vectors. Also, recall that if $R$ is row equivalent to $A$, then $R\vec{x} = \vec{0}$ has the same solutions as $A\vec{x} = \vec{0}$. This means that the columns of $R$ have *the same* dependency relationships as the columns of $A$.

**Example 3.47:** Find a basis for the column space of

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

**Solution:** The reduced row echelon form is

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Write $\vec{r}_i$ for the columns of $R$ and $\vec{a}_i$ for the columns of $A$. You can see immediately that $\vec{r}_3 = \vec{r}_1 + 2\vec{r}_2$ and $\vec{r}_5 = -\vec{r}_1 + 3\vec{r}_2 + 4\vec{r}_4$. So $\mathrm{col}(R) = \mathrm{span}(\vec{r}_1, \vec{r}_2, \vec{r}_4)$, and these three are linearly independent since they are standard unit vectors.

It follows that the columns of $A$ have the same dependency relationships: $\vec{a}_3 = \vec{a}_1 + 2\vec{a}_2$ and $\vec{a}_5 = -\vec{a}_1 + 3\vec{a}_2 + 4\vec{a}_4$. Also, $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_4$ must be linearly independent. So a basis for $\mathrm{col}(A)$ is given by $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_4$.

Note that these are the **columns corresponding to the leading 1's** in $R$!

**Warning:** Elementary row operations change the column space! So $\text{col}(A) \neq \text{col}(R)$. So while $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_4$ are a basis for $\text{col}(R)$, they are not a solution to the question asked.

We already saw that from $R$ we can read off a basis of $\text{row}(A)$. Since $\text{row}(A) = \text{row}(R)$, a basis for $\text{row}(A)$ consists of the non-zero rows of $R$.

The other kind of subspace that arises a lot is the **null space** of a matrix $A$, the subspace of solutions to the homogeneous system $A\vec{x} = \vec{0}$. We learned in Chapter 2 how to find a basis for this subspace, even though we didn't use this terminology.

**Example 3.48:** Find a basis for the null space of the $5 \times 4$ matrix $A$ above.

**Solution:** The reduced row echelon form of $[A \mid \vec{0}\,]$ is

$$[R \mid \vec{0}\,] = \left[\begin{array}{ccccr|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We see that $x_3$ and $x_5$ are free variables, so we let $x_3 = s$ and $x_5 = t$ and use back substitution to find that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \qquad \text{(See text.)}$$

Therefore, the two column vectors shown form a basis for the null space.

The vectors that arise in this way will always be linearly independent, since if all $x_i$'s are $0$, then the free variables must be zero, so the parameters must be zero.

## Summary

Finding bases for $\text{row}(A)$, $\text{col}(A)$ and $\text{null}(A)$:

1. Find the reduced row echelon form $R$ of $A$.
2. The nonzero rows form a basis for $\text{row}(A) = \text{row}(R)$.

3. The columns of $A$ that correspond to the columns of $R$ with leading 1's form a basis for $\mathrm{col}(A)$.
4. Use back substitution to solve $R\vec{x} = \vec{0}$; the vectors that arise are a basis for $\mathrm{null}(A) = \mathrm{null}(R)$.

You just need to do row reduction *once* to answer all three questions!

We have seen two ways to compute a basis of a span of a set of vectors. One is to make them the columns of a matrix, and the other is to make them the rows. The column method produces a basis using vectors from the original set. Both ways require about the same amount of work.

Similarly, if asked to find a basis for $\mathrm{row}(A)$, one could use the column method on $A^T$.

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