Math 1600A Lecture 20, Section 2, 25 Oct 2013

Announcements:

Read Section 3.6 for Monday. Work through recommended homework questions.

Tutorials: No tutorials next week!

We're more than halfway done the lectures! This is lecture 20 out of 37.

Office hour: Monday, 1:30-2:30, MC103B.
Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Partial review of Lectures 18 and 19:

Subspaces

Definition: A **subspace** of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that:
1. The zero vector $\vec{0}$ is in S.
2. S is **closed under addition**: If $\vec{u}$ and $\vec{v}$ are in S, then $\vec{u} + \vec{v}$ is in S.
3. S is **closed under scalar multiplication**: If $\vec{u}$ is in S and c is any scalar, then $c\,\vec{u}$ is in S.

Basis

Definition: A **basis** for a subspace S of $\mathbb{R}^n$ is a set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ such that:
1. $S = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k)$, and
2. $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent.

Subspaces associated with matrices

Definition: Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace $\mathrm{row}(A)$ of $\mathbb{R}^n$ spanned by the rows of A.
2. The **column space** of A is the subspace $\mathrm{col}(A)$ of $\mathbb{R}^m$ spanned by the columns of A.
3. The **null space** of A is the subspace $\mathrm{null}(A)$ of $\mathbb{R}^n$ consisting of the solutions to the system $A\,\vec{x} = \vec{0}$.

Theorem 3.20: Let A and R be row equivalent matrices. Then $\mathrm{row}(A) = \mathrm{row}(R)$.

Also, $\text{null}(A) = \text{null}(R)$. But elementary row operations change the column space! So $\text{col}(A) \neq \text{col}(R)$.

Theorem: If R is a matrix in row echelon form, then the nonzero rows of R form a basis for $\text{row}(R)$.

So if R is a row echelon form of A, then a basis for $\text{row}(A)$ is given by the nonzero rows of R.

Now, since $\text{null}(A) = \text{null}(R)$, the columns of R have the same dependency relationships as the columns of A.

It is easy to see that the pivot columns of R form a basis for $\text{col}(R)$, so the corresponding columns of A form a basis for $\text{col}(A)$.

We learned in Chapter 2 how to use R to find a basis for the **null space** of a matrix A, even though we didn't use this terminology.

Summary

Finding bases for $\text{row}(A)$, $\text{col}(A)$ and $\text{null}(A)$:

1. Find the reduced row echelon form R of A.
2. The nonzero rows of R form a basis for $\text{row}(A) = \text{row}(R)$.
3. The columns of A that correspond to the columns of R with leading 1's form a basis for $\text{col}(A)$.
4. Use back substitution to solve $R\vec{x} = \vec{0}$; the vectors that arise are a basis for $\text{null}(A) = \text{null}(R)$.

Row echelon form is in fact enough. Then you look at the columns with leading nonzero entries (the pivot columns).

These methods can be used to compute a basis for a subspace S spanned by some vectors $\vec{v}_1, \ldots, \vec{v}_k$.

The **row method**:

1. Form the matrix A whose rows are $\vec{v}_1, \ldots, \vec{v}_k$, so $S = \text{row}(A)$.
2. Reduce A to row echelon form R.
3. The nonzero rows of R will be a basis of $S = \text{row}(A) = \text{row}(R)$.

The **column method**:

1. Form the matrix A whose columns are $\vec{v}_1, \ldots, \vec{v}_k$, so $S = \text{col}(A)$.

2. Reduce A to row echelon form R.
3. The columns of A that correspond to the columns of R with leading entries form a basis for $S = \text{col}(A)$.

New material

Dimension and Rank

We have seen that a subspace has many bases. Have you noticed anything about the number of vectors in each basis?

Theorem 3.23: Let S be a subspace of $\mathbb{R}^n$. Then any two bases for S have the same number of vectors.

Idea of proof:
Suppose that $\{\vec{u}_1, \vec{u}_2\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ were both bases for S. We'll show that this is impossible, by showing that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent. Since $\{\vec{u}_1, \vec{u}_2\}$ is a basis, we can express each $\vec{v}_i$ in terms of the $\vec{u}_j$'s:

$$\begin{aligned}\vec{v}_1 &= a_{11}\,\vec{u}_1 + a_{21}\,\vec{u}_2\\ \vec{v}_2 &= a_{12}\,\vec{u}_1 + a_{22}\,\vec{u}_2\\ \vec{v}_3 &= a_{13}\,\vec{u}_1 + a_{23}\,\vec{u}_2\end{aligned}$$

Then

$$\begin{aligned}&c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + c_3\,\vec{v}_3\\ &= c_1(a_{11}\,\vec{u}_1 + a_{21}\,\vec{u}_2) + c_2(a_{12}\,\vec{u}_1 + a_{22}\,\vec{u}_2) + c_3(a_{13}\,\vec{u}_1 + a_{23}\,\vec{u}_2)\\ &= (c_1a_{11} + c_2a_{12} + c_3a_{13})\,\vec{u}_1 + (c_1a_{21} + c_2a_{22} + c_3a_{23})\,\vec{u}_2\end{aligned}$$

But the homogenous system

$$\begin{aligned}c_1a_{11} + c_2a_{12} + c_3a_{13} &= 0\\ c_1a_{21} + c_2a_{22} + c_3a_{23} &= 0\end{aligned}$$

has nontrivial solutions! (Why?) Therefore, we can find nontrivial c_1, c_2, c_3 such that

$$c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + c_3\,\vec{v}_3 = \vec{0} \qquad \square$$

A very similar argument works for the general case.

Definition: The number of vectors in a basis for a subspace S is called the **dimension** of S, denoted $\dim S$.

Example: $\dim \mathbb{R}^n = n$

Example: If S is a line through the origin in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\dim S = 1$

Example: If S is a plane through the origin in $\mathbb{R}^3$, then $\dim S = 2$

Example: If $S = \text{span}(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix})$, then $\dim S = 2$.

Example: Let A be the matrix from last class whose reduced row echelon form is

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then: $\dim \text{row}(A) = 3 \quad \dim \text{col}(A) = 3 \quad \dim \text{null}(A) = 2$

Note that $\dim \text{row}(A) = \text{rank}(A)$, since we defined the rank of A to be the number of nonzero rows in R. The above theorem shows that this number doesn't depend on how you row reduce A.

We call the dimension of the null space the **nullity** of A and write $\text{nullity}(A) = \dim \text{null}(A)$. This is what we called the "number of free variables" in Chapter 2.

From the way we find the basis for $\text{row}(A)$, $\text{col}(A)$ and $\text{null}(A)$, can you deduce any relationships between their dimensions?

Theorems 3.24 and 3.26: Let A be an $m \times n$ matrix. Then

$$\dim \text{row}(A) = \dim \text{col}(A) = \text{rank}(A) \quad \text{and} \quad \text{rank}(A) + \text{nullity}(A) = n.$$

Very important!

Questions:

True/false: for any A, $\text{rank}(A) = \text{rank}(A^T)$. True, since $\text{rank}(A) = \dim \text{row}(A) = \dim \text{col}(A^T) = \text{rank}(A^T)$.

True/false: if A is 2×5, then the nullity of A is 3. False. We know that $\text{rank}(A) \leq 2$ and $\text{rank}(A) + \text{nullity}(A) = 5$, so $\text{nullity}(A) \geq 3$ (and ≤ 5).

True/false: if A is 5×2, then $\text{nullity}(A) \geq 3$. False. $\text{rank}(A) + \text{nullity}(A) = 2$, so $\text{nullity}(A) = 0$, 1 or 2.

Example: Find the nullity of

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{bmatrix}$$

and of M^T. Any guesses? The rows of M are linearly independent, so the rank is 2, so the nullity is $7 - 2 = 5$. The rank of M^T is also 2, so the nullity is $2 - 2 = 0$.

For larger matrices, you would compute the rank by row reduction.

Fundamental Theorem of Invertible Matrices, Version 2

Theorem 3.27: Let A be an $n \times n$ matrix. The following are equivalent:
a. A is invertible.
b. $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.
c. $A\vec{x} = \vec{0}$ has only the trivial (zero) solution.
d. The reduced row echelon form of A is I_n.
f. $\text{rank}(A) = n$
g. $\text{nullity}(A) = 0$
h. The columns of A are linearly independent.
i. The columns of A span $\mathbb{R}^n$.
j. The columns of A are a basis for $\mathbb{R}^n$.

Proof: We saw that (a), (b), (c) and (d) are equivalent in Theorem 3.12. The new ones

(d) $\iff$ (f): the only square matrix in row echelon form with n nonzero rows is I_n.

(f) $\iff$ (g): follows from $\text{rank}(A) + \text{nullity}(A) = n$.

(c) $\iff$ (h): easy.

(i) $\implies$ (f) $\implies$ (b) $\implies$ (i): Explain.

(h) and (i) $\iff$ (j): Clear.

In fact, since $\text{rank}(A) = \text{rank}(A^T)$, we can add the following:

k. The rows of A are linearly independent.
l. The rows of A span $\mathbb{R}^n$.
m. The rows of A are a basis for $\mathbb{R}^n$.

Example 3.52: Show that the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

Solution: Show that matrix A with these vectors as the columns has rank 3. On whiteboard.

Not covering Theorem 3.28.

Coordinates

Suppose S is a subspace of $\mathbb{R}^n$ with a basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$, so S has dimension k. Then we can assign **coordinates** to vectors in S, using the following theorem:

Theorem 3.29: For every vector v in S, there is *exactly one way* to write v as a linear combination of the vectors in $\mathcal{B}$:

$$\vec{v} = c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k$$

Proof: Try to work it out yourself! It's a good exercise.

We call the coefficients $c_1, c_2, \ldots, c_k$ the **coordinates of $\vec{v}$ with respect to $\mathcal{B}$**, and write

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

Example: Let S be the plane through the origin in $\mathbb{R}^3$ spanned by $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, so $\mathcal{B} = \{\, \vec{v}_1,\, \vec{v}_2 \}$ is a basis for S. Let $\vec{v} = \begin{bmatrix} 6 \\ 9 \\ 12 \end{bmatrix}$. Then

$$\vec{v} = 2\,\vec{v}_1 + 1\,\vec{v}_2 \qquad \text{so} \qquad [\,\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Note that while $\vec{v}$ is a vector in $\mathbb{R}^3$, it only has **two** coordinates with respect to $\mathcal{B}$.

Example: Let $\mathcal{B} = \{\, \vec{e}_1,\, \vec{e}_2,\, \vec{e}_3 \}$ be the standard basis for $\mathbb{R}^3$, and consider $\vec{v} = \begin{bmatrix} 6 \\ 9 \\ 12 \end{bmatrix}$. Then

$$\vec{v} = 6\,\vec{e}_1 + 9\,\vec{e}_2 + 12\,\vec{e}_3 \qquad \text{so} \qquad [\,\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 9 \\ 12 \end{bmatrix}$$

We've implicitly been using the standard basis everywhere, but often in applications it is better to use a basis suited to the problem.

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