Math 1600A Lecture 24, Section 2, 6 Nov 2013

Announcements:

Read Section 4.2 for Friday. Work through recommended homework questions.

Midterm 2: this Thursday evening, 7-8:30 pm. People with a **conflict** should already have let me know, and should know when the make-up is. Midterm 2 covers from Section 2.3 until the end of Chapter 3 (Wednesday), but builds on the earlier material as well. A **practice exam** is available from the course home page. Last name A-Q must write in **NS1**, R-Z in **NS7**. See the missed exam section of the course web page for policies, including for illness.

Tutorials: No quiz; focused on midterm review.

Office hour: today, 12:30-1:30, MC103B.
Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Exercises for Appendix C are here, and there are solutions.

New Material: Section 4.1: Eigenvalues and eigenvectors

We saw when studying Markov chains that it was important to find solutions to the system $A\vec{x} = \vec{x}$, where A is a square matrix. We did this by solving $(I - A)\vec{x} = \vec{0}$.

More generally, a central problem in linear algebra is to find $\vec{x}$ such that $A\vec{x}$ is a scalar multiple of $\vec{x}$.

Definition: Let A be an $n \times n$ matrix. A scalar λ (lambda) is called an **eigenvalue** of A if there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Such a vector $\vec{x}$ is called an **eigenvector** of A corresponding to λ.

We showed that $\lambda = 1$ is an eigenvalue of every stochastic matrix A.

Example: Since

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

we see that 2 is an eigenvalue of $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ with eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Example 4.2: Show that 5 is an eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and determine all eigenvectors corresponding to this eigenvalue.

Solution: We are looking for nonzero solutions to $A\,\vec{x} = 5\,\vec{x}$. This is the same as $(A - 5I)\,\vec{x} = \vec{0}$, so we compute the coefficient matrix:

$$A - 5I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$$

The columns are linearly dependent, so the null space of $A - 5I$ is nonzero. So $A\,\vec{x} = 5\,\vec{x}$ has a nontrivial solution, which is what it means for 5 to be an eigenvalue.

To find the eigenvectors, we compute the null space of $A - 5I$:

$$[\,A - 5I \mid \vec{0}\,] = \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solutions are of the form $\begin{bmatrix} t/2 \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$. So the eigenvectors for the eigenvalue 5 are the *nonzero* multiples of $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$.

Definition: Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is a subspace called the **eigenspace** of λ and is denoted E_λ. In other words,

$$E_\lambda = \mathrm{null}(A - \lambda I).$$

In the above Example, $E_5 = \mathrm{span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$.

Example: Give an eigenvalue of the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and compute its eigenspace.

Since $A\,\vec{x} = 2\,\vec{x}$ for every $\vec{x}$, 2 is an eigenvalue, and is the only eigenvalue. In this case, $E_2 = \mathbb{R}^2$.

Example: If 0 is an eigenvalue of A, what is another name for E_0?

E_0 is the null space of $A - 0I = A$. That is, $E_0 = \mathrm{null}(A)$.

Applet: This java applet lets you search for eigenvectors. (Instructions.)

Try it with:

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

(If that doesn't work, here is another applet.)

See Pages 268 and 269 of the text for another geometric way to understand eigenvalues and eigenvectors (Figure 4.7).

Read Example 4.3 in the text for a 3×3 example.

Finding eigenvalues

Given a specific number λ, we now know how to check whether λ is an eigenvalue: we check whether $A - \lambda I$ has a nontrivial null space. And we can find the eigenvectors by finding the null space.

We also have a geometric way to find **all** eigenvalues λ, at least in the 2×2 case. Is there an algebraic way to check all λ at once?

By the fundamental theorem of invertible matrices, $A - \lambda I$ has a nontrivial null space if and only if it is not invertible. For 2×2 matrices, we can check invertibility using the determinant!

Example: Find all eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

Solution: We need to find all λ such that $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} = (1-\lambda)(-2-\lambda) - 4 = \lambda^2 + \lambda$$

so we need to solve the quadratic equation $\lambda^2 + \lambda - 6 = 0$. This can be factored as $(\lambda - 2)(\lambda + 3) = 0$, and so $\lambda = 2$ or $\lambda = -3$, the same as we saw above and with the applet.

We could proceed to find the eigenvectors for these eigenvalues, by solving $(A - 2)\,\vec{x} = \vec{0}$ and $(A + 3)\,\vec{x} = \vec{0}$. Do this on whiteboard, if time.

Appendix D provides review of polynomials and their solutions.

See also Example 4.5 in text.

The eigenvalues depend on whether you let your vectors have coefficients in $\mathbb{R}$ or in $\mathbb{C}$:

Example 4.7: Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (a) over $\mathbb{R}$ and (b) over $\mathbb{C}$.

Solution: We must solve

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1.$$

(a) Over $\mathbb{R}$, there are no solutions, so A has no real eigenvalues. (See the applet above, with its default matrix.)

(b) Over $\mathbb{C}$, the solutions are $\lambda = i$ and $\lambda = -i$. The eigenvectors for $\lambda = i$ are the nonzero multiples of $\begin{bmatrix} i \\ 1 \end{bmatrix}$, since

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

So now we know how to handle the 2×2 case. To handle larger matrices, we need to learn about their determinants, which is Section 4.2.

We won't discuss eigenvectors and eigenvalues for matrices over $\mathbb{Z}_m$.