Math 1600A Lecture 25, Section 2, 8 Nov 2013

Announcements:

Continue **reading** Section 4.2 for Monday. Work through recommended homework questions.

Tutorials: No quiz next week, just review.
Office hour: Monday, 1:30-2:30, MC103B.
Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Brief summary of Section 4.1: Eigenvalues and eigenvectors

Definition: Let A be an $n \times n$ matrix. A scalar λ (lambda) is called an **eigenvalue** of A if there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Such a vector $\vec{x}$ is called an **eigenvector** of A corresponding to λ.

Question: Why do we only consider *square* matrices here?

Example: Since

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

we see that 2 is an eigenvalue of $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ with eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

In general, the eigenvectors for a given eigenvalue λ are the nonzero solutions to $(A - \lambda I)\vec{x} = \vec{0}$.

Definition: Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is a subspace called the **eigenspace** of λ and is denoted E_λ. In other words,

$$E_\lambda = \mathrm{null}(A - \lambda I).$$

We worked out many examples, and used an applet to understand the geometry.

Finding eigenvalues

Given a specific number λ, we know how to check whether λ is an eigenvalue: we check whether $A - \lambda I$ has a nontrivial null space. (And we can find the eigenvectors by finding the null space.)

By the fundamental theorem of invertible matrices, $A - \lambda I$ has a nontrivial null space if and only if it is not invertible. For 2×2 matrices, we can check invertibility using the determinant!

Example: Find all eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

Solution: We need to find all λ such that $\det(A - \lambda I) = 0$.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 2 \\ 2 & -2 - \lambda \end{bmatrix} = (1 - \lambda)(-2 - \lambda) - 4 = \lambda^2 + \lambda - 6,$$

so we need to solve the quadratic equation $\lambda^2 + \lambda - 6 = 0$. This can be factored as $(\lambda - 2)(\lambda + 3) = 0$, and so $\lambda = 2$ or $\lambda = -3$ are the eigenvalues.

So now we know how to handle the 2×2 case. To handle larger matrices, we need to learn about their determinants, which is Section 4.2.

New material: Section 4.2: Determinants

Recall that we defined the determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by $\det A = ad - bc$. We also write this as

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a 3×3 matrix A, we define

$$\det A = \begin{vmatrix} \color{red}{a_{11}} & \color{red}{a_{12}} & \color{red}{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \color{red}{a_{11}} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \color{red}{a_{12}} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \color{red}{a_{13}} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

If we write A_{ij} for the matrix obtained from A by deleting the ith row and the jth column, then this can be written

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} = \sum_{j=1}^{3} (-1)^{1+j} \, a_{1j} \det A_{1j}.$$

We call $\det A_{ij}$ the (i, j)**-minor** of A.

Example: On whiteboard.

Example 4.9 in the book shows another method, that doesn't generalize to larger matrices.

Determinants of $n \times n$ matrices

Definition: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then the **determinant** of A is the scalar

$$\begin{aligned} \det A = |A| &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} \, a_{1j} \det A_{1j}. \end{aligned}$$

This is a recursive definition!

Example: $A = \begin{vmatrix} 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 3 \\ 2 & 0 & 4 & 5 \end{vmatrix}$, on whiteboard.

The computation can be very long if there aren't many zeros! We'll learn some better methods.

Note that if we define the determinant of a 1×1 matrix $A = [a]$ to be a, then the general definition works in the 2×2 case as well. So, in this context, $|a| = a$ (not the absolute value!)

It will make the notation simpler if we define the (i, j)**-cofactor** of A to be

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Then the definition above says

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}.$$

This is called the **cofactor expansion along the first row**. It turns out that *any* row or column works!

Theorem 4.1 (The Laplace Expansion Theorem): Let A be any $n \times n$ matrix. Then for each i we have

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij} C_{ij}$$

(**cofactor expansion along the ith row**). And for each j we have

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij} C_{ij}$$

(**cofactor expansion along the jth column**).

The book proves this result at the end of this section, but we won't cover the proof.

The signs in the cofactor expansion form a checkerboard pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example: Redo the previous 4×4 example, saving work by expanding along the second column. On whiteboard. Note that the $+-$ pattern for the 3×3 determine is not from the original matrix.

Example: A 4×4 triangular matrix, on whiteboard.

A **triangular** matrix is a square matrix that is all zero below the diagonal or above the diagonal.

Theorem 4.2: If A is triangular, then $\det A$ is the product of the diagonal entries.

Better methods

Laplace Expansion is convenient when there are appropriately placed zeros in the matrix, but it is not good in general. It produces $n!$ different terms (explain). A supercomputer would require 10^{30} times the age of the universe just to compute a 50×50 determinant in this way. And that's a puny determinant for real-world applications.

So how do we do better? Like always, we turn to row reduction! These properties will be what we need:

Theorem 4.3: Let A be a square matrix.

a. If A has a zero row, the $\det A = 0$.

b. If B is obtained from A by interchanging two rows, then $\det B = -\det A$.
c. If A has two identical rows, then $\det A = 0$.
d. If B is obtained from A by multiplying a row of A by k, then $\det B = k \det A$.
e. If A, B and C are identical in all rows except the ith row, and the ith row of C is the sum of the ith rows of A and B, then $\det C = \det A + \det B$.
f. If B is obtained from A by adding a multiple of one row to another, then $\det B = \det A$.

All of the above statements are true with rows replaced by columns.

Explain verbally, making use of:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

The following will help explain how (f) follows from (d) and (e):

$$A = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \\ \vec{r}_4 \end{bmatrix}, \quad B = \begin{bmatrix} \vec{r}_1 \\ 5\,\vec{r}_4 \\ \vec{r}_3 \\ \vec{r}_4 \end{bmatrix}, \quad B' = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_4 \\ \vec{r}_3 \\ \vec{r}_4 \end{bmatrix}, \quad C = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 + 5\,\vec{r}_4 \\ \vec{r}_3 \\ \vec{r}_4 \end{bmatrix}.$$

$$\det C = \det A + \det B = \det A + 5 \det B' = \det A + 5(0) = \det A.$$

The bold statements are the ones that are useful for understanding how row operations change the determinant.