Math 1600A Lecture 27, Section 2, 13 Nov 2013

Announcements:

Today we finish 4.2 and start 4.3. Continue **reading** Section 4.3 for Friday and also read **Appendix D** on polynomials (**self-study**).
Work through recommended homework questions.

Tutorials: No quiz this week, just review.
Office hour: Wednesday, 12:30-1:30, MC103B.
Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Midterm 2 Solutions are available from the course home page. I don't know the class average yet.

Review Questions

Question: True/false: If A is not invertible, then AB is not invertible.

Question: True/false: $\det(A+B) = \det A + \det B$.

Question: $\det(3I_2) = 3^2 \det I_2 = 3^2 = 9$

Question: $\begin{vmatrix} 0 & 0 & a \\ 0 & b & c \\ d & e & f \end{vmatrix} = - \begin{vmatrix} d & e & f \\ 0 & b & c \\ 0 & 0 & a \end{vmatrix} = -abd \quad \text{(not triangular!)}$

Partial review of last class: Cramer's Rule

Notation: If A is an $n \times n$ matrix and $\vec{b} \in \mathbb{R}^n$, we write $A_i(\vec{b})$ for the matrix obtained from A by replacing the ith column with the vector $\vec{b}$:

$$A_i(\vec{b}) = [\, \vec{a}_1 \cdots \vec{a}_{i-1}\, \vec{b}\, \vec{a}_{i+1} \cdots \vec{a}_n \,]$$

Theorem: Let A be an invertible $n \times n$ matrix and let $\vec{b}$ be in $\mathbb{R}^n$. Then the unique solution $\vec{x}$ of the system $A\vec{x} = \vec{b}$ has components

$$x_i = \frac{\det(A_i(\vec{b}))}{\det A}\,, \quad \text{for } i = 1, \ldots, n$$

New material: Matrix Inverse using the Adjoint

Suppose A is invertible. We'll use Cramer's rule to find a formula for $X = A^{-1}$. We know that $AX = I$, so the jth column of X satisfies $A\vec{x}_j = \vec{e}_j$. By Cramer's Rule,

$$x_{ij} = \frac{\det(A_i(\vec{e}_j))}{\det A}$$

By expanding along the ith column, we see that

$$\det(A_i(\vec{e}_j)) = C_{ji}$$

So

$$x_{ij} = \frac{1}{\det A} C_{ji}, \quad \text{i.e.,} \quad X = \frac{1}{\det A} [C_{ij}]^T$$

The matrix

$$\text{adj}A := [C_{ji}] = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

is called the **adjoint** of A.

Theorem: If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det A} \text{adj}A$$

Example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the cofactors are

$$\begin{aligned} C_{11} &= +\det[d] = +d & \qquad C_{12} &= -\det[c] = -c \\ C_{21} &= -\det[b] = -b & \qquad C_{22} &= +\det[a] = +a \end{aligned}$$

so the adjoint matrix is

$$\text{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{\det A} \text{adj} A = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

as we saw before.

See Example 4.17 in the text for a 3×3 example. Again, this is not generally a good computational approach. It's importance is theoretical.

Section 4.3: Eigenvalues and Eigenvectors

Recall from Section 4.1:

Definition: Let A be an $n \times n$ matrix. A scalar λ (lambda) is called an **eigenvalue** of A if there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Such a vector $\vec{x}$ is called an **eigenvector** of A corresponding to λ.

The eigenvectors for a given eigenvalue λ are the **nonzero** solutions to $(A - \lambda I)\vec{x} = \vec{0}$.

Definition: The collection of **all** solutions to $(A - \lambda I)\vec{x} = \vec{0}$ is a subspace called the **eigenspace** of λ and is denoted E_λ. In other words,

$$E_\lambda = \text{null}(A - \lambda I).$$

It consists of the eigenvectors plus the zero vector.

By the fundamental theorem of invertible matrices, $A - \lambda I$ has a nontrivial null space if and only if it is not invertible, and we now know that this is the case if and only if $\det(A - \lambda I) = 0$.

The expression $\det(A - \lambda I)$ is always a polynomial in λ. For example, when

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad - b$$

In A is 3×3, then

$$\det(A - \lambda I) = (a_{11} - \lambda)\begin{vmatrix} a_{22}-\lambda & a_{23} \\ a_{32} & a_{33}-\lambda \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}-\lambda \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} - \\ a_{31} & a_3 \end{vmatrix}$$

which is a degree 3 polynomial in λ.

Similarly, if A is $n \times n$, $\det(A - \lambda I)$ will be a degree n polynomial in λ. It is called the **characteristic polynomial** of A, and $\det(A - \lambda I) = 0$ is called the **characteristic equation**.

Finding eigenvalues and eigenspaces: Let A be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$.
2. Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$.
3. For each eigenvalue λ, find a basis for $E_\lambda = \text{null}(A - \lambda I)$ by solving the system $(A - \lambda I)\,\vec{x} = \vec{0}$.

So we need to get good at solving polynomial equations. Solutions are called **zeros** or **roots**.

Theorem D.4 (The Fundamental Theorem of Algebra): A polynomial of degree n has at most n distinct roots.

Therefore:

Theorem: An $n \times n$ matrix A has at most n distinct eigenvalues.

Also:

Theorem D.2 (The Factor Theorem): Let f be a polynomial and let a be a constant. Then a is a zero of $f(x)$ (i.e. $f(a) = 0$) if and only if $x - a$ is a factor of $f(x)$ (i.e. $f(x) = (x-a)g(x)$ for some polynomial g).

Example 4.18: Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$.

Solution: 1. On whiteboard, compute the characteristic polynomial:

$$\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$$

2. To find the roots, it is often worth trying a few small integers to start. We see that $\lambda = 1$ works. So by the factor theorem, we know $\lambda - 1$ is a factor:

$$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = (\lambda - 1)(?\lambda^2 + ?\lambda + ?)$$

Now we need to find roots of $-\lambda^2 + 3\lambda - 2$. Again, $\lambda = 1$ works, and this factors as $-(\lambda - 1)(\lambda - 2)$. So

$$\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2)$$

and the roots are $\lambda = 1$ and $\lambda = 2$.

3. To find the $\lambda = 1$ eigenspace, we do row reduction:

$$[A - I \mid 0] = \left[\begin{array}{rrr|r} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We find that $x_3 = t$ is free and $x_1 = x_2 = x_3$, so

$$E_1 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

So $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis of the eigenspace corresponding to $\lambda = 1$.

Finding a basis for E_2 is similar; see text.

A root a of a polynomial f implies that $f(x) = (x - a)g(x)$. Sometimes, a is also a root of $g(x)$, as we found above. Then $f(x) = (x - a)^2 h(x)$. The largest k such that $(x - a)^k$ is a factor of f is called the **multiplicity** of the root a in f.

In the case of an eigenvalue, we call its multiplicity in the characteristic polynomial the **algebraic multiplicity** of this eigenvalue.

In the previous example, $\lambda = 1$ has algebraic multiplicity 2 and $\lambda = 2$ has algebraic multiplicity 1.

We also define the **geometric multiplicity** of an eigenvalue λ to be the dimension of the corresponding eigenspace. In the previous example, $\lambda = 1$ has geometric multiplicity 1 (and so does $\lambda = 2$).

Example 4.19: Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$. Do partially, on whiteboard.

In this case, we find that $\lambda = 0$ has algebraic multiplicity 2 and geometric multiplicity 2.

These multiplicities will be important in Section 4.4.