Math 1600A Lecture 28, Section 2, 15 Nov 2013

Announcements:

Today we finish 4.3. **Read** Section 4.4 for Monday and also read **Appendix D** on polynomials (**self-study**). Work through recommended homework questions.

Tutorials: Quiz 5 is next week.
Office hour: Monday, 1:30-2:30, MC103B.
Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Midterm 2 Solutions are available from the course home page. The average was 27/40 = 68%.

Question: If P is invertible, how do $\det A$ and $\det(P^{-1}AP)$ compare?

They are equal:
$\det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)}\det(A)\det(P) = \det A$.

Partial review of last class: Section 4.3

Definition: Let A be an $n \times n$ matrix. A scalar λ (lambda) is called an **eigenvalue** of A if there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$, i.e. $(A - \lambda I)\vec{x} = \vec{0}$. Such a vector $\vec{x}$ is called an **eigenvector** of A corresponding to λ.

Definition: The collection of **all** solutions to $(A - \lambda I)\vec{x} = \vec{0}$ is a subspace called the **eigenspace** of λ and is denoted E_λ. In other words,

$$E_\lambda = \text{null}(A - \lambda I).$$

It consists of the eigenvectors plus the zero vector.

Definition: If A is $n \times n$, $\det(A - \lambda I)$ will be a degree n polynomial in λ. It is called the **characteristic polynomial** of A, and $\det(A - \lambda I) = 0$ is called the **characteristic equation**.

By the fundamental theorem of invertible matrices, the solutions to the characteristic equation are exactly the eigenvalues.

Finding eigenvalues and eigenspaces: Let A be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$.
2. Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$.
3. For each eigenvalue λ, find a basis for $E_\lambda = \text{null}(A - \lambda I)$ by solving the system

$(A - \lambda I)\,\vec{x} = \vec{0}$.

So we need to get good at solving polynomial equations. Solutions are called **zeros** or **roots**.

Theorem D.4 (The Fundamental Theorem of Algebra): A polynomial of degree n has at most n distinct roots.

Therefore:

Theorem: An $n \times n$ matrix A has at most n distinct eigenvalues.

Also:

Theorem D.2 (The Factor Theorem): Let f be a polynomial and let a be a constant. Then a is a zero of $f(x)$ (i.e. $f(a) = 0$) if and only if $x - a$ is a factor of $f(x)$ (i.e. $f(x) = (x - a)g(x)$ for some polynomial g).

The largest k such that $(x - a)^k$ is a factor of f is called the **multiplicity** of the root a in f.

In the case of an eigenvalue, we call its multiplicity in the characteristic polynomial the **algebraic multiplicity** of this eigenvalue.

We also define the **geometric multiplicity** of an eigenvalue λ to be the dimension of the corresponding eigenspace E_λ.

New material: 4.3 continued

Theorem 4.15: The eigenvalues of a triangular matrix are the entries on its main diagonal (repeated according to their algebraic multiplicity).

Example: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 1 \end{bmatrix}$, then

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 0 \\ 4 & 5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda),$$

so the eigenvalues are $\lambda = 1$ (with algebraic multiplicity 2) and $\lambda = 3$ (with algebraic multiplicity 1).

Question: What are the eigenvalues of a diagonal matrix?

The eigenvalues are the diagonal entries.

Question: What are the eigenvalues of $\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$?

The characteristic polynomial is

$$\begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2),$$

so the eigenvalues are 2 and -2. Trick question.

Question: How can we tell whether a matrix A is invertible using eigenvalues?

A is invertible if and only if 0 is not an eigenvalue, because 0 being an eigenvalue is equivalent to $\mathrm{null}(A)$ being non-trivial, which is equivalent to A not being invertible, by the fundamental theorem.

So we can extend the fundamental theorem with two new entries:

Theorem 4.17: Let A be an $n \times n$ matrix. The following are equivalent:
a. A is invertible.
b. $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.
c. $A\vec{x} = \vec{0}$ has only the trivial (zero) solution.
d. The reduced row echelon form of A is I_n.
f. $\mathrm{rank}(A) = n$
g. $\mathrm{nullity}(A) = 0$
h. The columns of A are linearly independent.
i. The columns of A span $\mathbb{R}^n$.
j. The columns of A are a basis for $\mathbb{R}^n$.
k. The rows of A are linearly independent.
l. The rows of A span $\mathbb{R}^n$.
m. The rows of A are a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of A

Eigenvalues of powers and inverses

Suppose $\vec{x}$ is an eigenvector of A with eigenvalue λ. What can we say about A^2? A^3? If A is invertible, how about the eigenvalues/vectors of A^{-1}? On whiteboard.

We've shown:

Theorem 4.18: If $\vec{x}$ is an eigenvector of A with eigenvalue λ, then $\vec{x}$ is an eigenvector of A^k with eigenvalue λ^k. This holds for each integer $k \geq 0$, and also for $k < 0$ if A is invertible.

In contrast to some other recent results, this one is very useful computationally:

Example 4.21: Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Solution: By finding the eigenspaces of the matrix, we can show that

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Write $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Since $\vec{x} = 3\,\vec{v}_1 + 2\,\vec{v}_2$ we have

$$\begin{aligned} A^{10}\,\vec{x} &= A^{10}(3\,\vec{v}_1 + 2\,\vec{v}_2) = 3A^{10}\,\vec{v}_1 + 2A^{10}\,\vec{v}_2 \\ &= 3(-1)^{10}\,\vec{v}_1 + 2(2^{10})\,\vec{v}_2 = \begin{bmatrix} 3 + 2^{11} \\ -3 + 2^{12} \end{bmatrix} \end{aligned}$$

Much faster than repeated matrix multiplication, especially if 10 is replaced with 100.

This raises an interesting question. In the example, the eigenvectors were a basis for $\mathbb{R}^2$, so we could use this method to compute $A^k\,\vec{x}$ for any $\vec{x}$. However, last class we saw a 3×3 matrix with two one-dimensional eigenspaces, so the eigenvectors didn't span $\mathbb{R}^3$. We will study this further in Section 4.4, but right now we can answer a related question about linear independence.

Theorem: If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly independent.

Proof in case $m = 2$: If $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent, then $\vec{v}_1 = c\,\vec{v}_2$ for some c. Therefore

$$A\,\vec{v}_1 = A\,c\,\vec{v}_2 = cA\,\vec{v}_2$$

so

$$\lambda_1\,\vec{v}_1 = c\lambda_2\,\vec{v}_2 = \lambda_2\,\vec{v}_1$$

Since $\vec{v}_1 \neq \vec{0}$, this forces $\lambda_1 = \lambda_2$, a contradiction. $\square$

The general case is very similar; see text.

If time: how to become a Billionaire using the material from this course.