# Math 1600A Lecture 29, Section 2, 18 Nov 2013

## Announcements:

Today we start 4.4. Continue **reading** Section 4.4 for Wednesday. Work through recommended homework questions.

**Tutorials:** Quiz 5 is this week. It covers Appendix C, 4.1, 4.2 and 4.3
**Office hour:** Monday, 1:30-2:30, MC103B.
**Help Centers:** Monday-Friday 2:30-6:30 in MC 106.
**Course evaluations** will be at the start of Wednesday's class.

The **final exam** will take place on Mon Dec 9, 2-5pm.
Section 001: HSB 236 (last names A-W), HSB 240 (last names X-Z)
Section 002: HSB 240 (last names A-**LI**), HSB 35 (last names LIU-Z)
The final exam will cover all the material from the course, but will emphasize the later material. See the course home page for final exam **conflict** policy. You should **already** have notified the registrar or your Dean (and me) of any conflicts!

### Partial review of Section 4.3

The eigenvalues of a square matrix $A$ can be computed as the **roots** (also called **zeros**) of the **characteristic polynomial**

$$\det(A - \lambda I)$$

**Theorem D.2 (The Factor Theorem):** Let $f$ be a polynomial and let $a$ be a constant. Then $a$ is a root of $f(x)$ (i.e. $f(a) = 0$) if and only if $x - a$ is a factor of $f(x)$ (i.e. $f(x) = (x-a)g(x)$ for some polynomial $g$).

The largest $k$ such that $(x-a)^k$ is a factor of $f$ is called the **multiplicity** of the root $a$ in $f$.

**Example:** Let $f(x) = x^2 - 2x + 1$. Since $f(1) = 1 - 2 + 1 = 0$, $1$ is a root of $f$. And since $f(x) = (x-1)^2$, $1$ has multiplicity $2$.

In the case of an eigenvalue, we call its multiplicity in the characteristic polynomial the **algebraic multiplicity** of this eigenvalue.

We also define the **geometric multiplicity** of an eigenvalue $\lambda$ to be the dimension of the corresponding eigenspace $E_\lambda$.

**Theorem 4.15:** The eigenvalues of a triangular matrix are the entries on its main diagonal (repeated according to their algebraic multiplicity).

**Theorem D.4 (The Fundamental Theorem of Algebra):** A polynomial of degree $n$ has at most $n$ distinct roots. In fact, the sum of the multiplicities is at most $n$.

Therefore:

**Theorem:** An $n \times n$ matrix $A$ has at most $n$ distinct eigenvalues. In fact, the sum of the algebraic multiplicities is at most $n$.

## New material: complex eigenvalues and eigenvectors

This material isn't covered in detail in the text.

**Example 4.7:** Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (a) over $\mathbb{R}$ and (b) over $\mathbb{C}$.

**Solution:** We must solve

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1.$$

(a) Over $\mathbb{R}$, there are no solutions, so $A$ has no real eigenvalues. This is why the Theorem above says "at most $n$".

(b) Over $\mathbb{C}$, the solutions are $\lambda = i$ and $\lambda = -i$. For example, the eigenvectors for $\lambda = i$ are the nonzero **complex** multiples of $\begin{bmatrix} i \\ 1 \end{bmatrix}$, since

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

In fact, $\lambda^2 + 1 = (\lambda - i)(\lambda + i)$, so each of these eigenvalues has algebraic multiplicity 1. So in this case the sum of the algebraic multiplicities is **exactly** 2.

The Fundamental Theorem of Algebra can be extended to say:

**Theorem D.4 (The Fundamental Theorem of Algebra):** A polynomial of degree $n$ has at most $n$ distinct **complex** roots. In fact, the sum of their multiplicities is **exactly** $n$.

Another way to put it is that over the complex numbers, every polynomial factors into **linear** factors.

## Real matrices

Notice that $i$ and $-i$ are complex conjugates of each other.

If the matrix $A$ has only real entries, then the characteristic polynomial has real coefficients. Say it is

$$\det(A - \lambda I) = a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0,$$

with all of the $a_i$'s real numbers. If $\alpha$ is an eigenvalue, then so is its complex conjugate $\bar{\alpha}$, because

$$\begin{aligned} & a_n \bar{\alpha}^n + a_{n-1}\bar{\alpha}^{n-1} + \cdots + a_1 \bar{\alpha} + a_0 \\ & \qquad = \overline{a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha + a_0} = \bar{0} = 0. \end{aligned}$$

**Theorem:** The complex eigenvalues of a **real** matrix come in conjugate pairs.

## Complex matrices

A complex matrix might have real or complex eigenvalues, and the complex eigenvalues do not have to come in conjugate pairs.

**Examples:** $\begin{bmatrix} 1 & 2 \\ 0 & i \end{bmatrix}$, $\begin{bmatrix} 1 & i \\ 0 & 2 \end{bmatrix}$.

## General case

In general, don't forget that the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives the roots of $ax^2 + bx + c$, and these can be real (if $b^2 - 4ac \geqslant 0$) or complex (if $b^2 - 4ac < 0$).

And try small integers first.

**Example:** Find the real and complex eigenvalues of $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

**Solution:**

$$\begin{vmatrix} 2-\lambda & 3 & 0 \\ 1 & 2-\lambda & 2 \\ 0 & -2 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 0 & 1-\lambda \end{vmatrix}$$
$$= (2-\lambda)(\lambda^2 - 3\lambda + 6) - 3(1-\lambda)$$
$$= -\lambda^3 + 5\lambda^2 - 9\lambda + 9.$$

By trial and error, $\lambda = 3$ is a root. So we factor:

$$-\lambda^3 + 5\lambda^2 - 9\lambda + 9 = (\lambda - 3)(-\lambda^2 + 2\lambda - 3)$$

We don't find any obvious roots for the quadratic factor, so we use the quadratic formula:

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(-1)(-3)}}{-2} = \frac{-2 \pm \sqrt{-8}}{-2}$$
$$= \frac{-2 \pm 2\sqrt{2}\,i}{-2} = 1 \pm \sqrt{2}\,i.$$

So the eigenvalues are $3$, $1 + \sqrt{2}\,i$ and $1 - \sqrt{2}\,i$.

**Note:** Our questions always involve real eigenvalues and real eigenvectors unless we say otherwise. But there **will** be problems where we ask for complex eigenvalues.

## More review: Eigenvalues of powers and inverses

**Theorem 4.18:** If $\vec{x}$ is an eigenvector of $A$ with eigenvalue $\lambda$, then $\vec{x}$ is an eigenvector of $A^k$ with eigenvalue $\lambda^k$. This holds for each integer $k \geq 0$, and also for $k < 0$ if $A$ is invertible.

We saw that this was useful computationally. We also saw:

**Theorem:** If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are eigenvectors of $A$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly independent.

We saw that sometimes the eigenvectors span $\mathbb{R}^n$, and sometimes they don't.

## Section 4.4: Similarity and Diagonalization

We're going to introduce a new concept that will turn out to be closely related to eigenvalues and eigenvectors.

**Definition:** Let $A$ and $B$ be $n \times n$ matrices. We say that $A$ is **similar** to $B$ if there is an invertible matrix $P$ such that $P^{-1}AP = B$. When this is the case, we write $A \sim B$.

It is equivalent to say that $AP = PB$ or $A = PBP^{-1}$.

**Example 4.22:** Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$. Then $A \sim B$, since

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}.$$

We also need to check that the matrix $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is invertible, which is the case since its determinant is $2$.

It is tricky in general to find such a $P$ when it exists. We'll learn a method that works in a certain situation in this section.

**Theorem 4.21:** Let $A$, $B$ and $C$ be $n \times n$ matrices. Then:
a. $A \sim A$.
b. If $A \sim B$ then $B \sim A$.
c. If $A \sim B$ and $B \sim C$, then $A \sim C$.

**Proof:** (a) $I^{-1}AI = A$

(b) Suppose $A \sim B$. Then $P^{-1}AP = B$ for some invertible matrix $P$. Then $PBP^{-1} = A$. Let $Q = P^{-1}$. Then $Q^{-1}BQ = A$, so $B \sim A$.

(c) Exercise. $\square$

Similar matrices have a lot of properties in common.

**Theorem 4.22:** Let $A$ and $B$ be similar matrices. Then:
a. $\det A = \det B$
b. $A$ is invertible iff $B$ is invertible.
c. $A$ and $B$ have the same rank.
d. $A$ and $B$ have the same characteristic polynomial.
e. $A$ and $B$ have the same eigenvalues.

**Proof:** Assume that $P^{-1}AP = B$ for some invertible matrix $P$.

We discussed (a) last time:

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P)$$
$$= \frac{1}{\det(P)}\det(A)\det(P) = \det A.$$

(b) follows immediately.

(c) takes a bit of work and will not be covered.

(d) follows from (a): since $B - \lambda I = P^{-1}AP - \lambda I = P^{-1}(A - \lambda I)P$ it follows that $B - \lambda I$ and $A - \lambda I$ have the same determinant.

(e) follows from (d). $\square$

**Question:** Are $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ similar?

**Question:** Are $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ similar?

See also Example 4.23(b) in text.

## Diagonalization

**Definition:** $A$ is **diagonalizable** if it is similar to some diagonal matrix.

**Example 4.24:** $A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$ is diagonalizable. Take $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$. Then

$$P^{-1}AP = \cdots = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

If $A$ is similar to a diagonal matrix $D$, then $D$ must have the eigenvalues of $A$ on the diagonal. But how to find $P$?

**Theorem 4.23:** Let $A$ be an $n \times n$ matrix. Then $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.

More precisely, there exist an invertible matrix $P$ and a diagonal matrix $D$ with $P^{-1}AP = D$ if and only if the columns of $P$ are $n$ linearly independent eigenvectors of $A$ and the diagonal entries of $D$ are the corresponding eigenvalues in the same order.

**Proof:** Suppose $\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_n$ are $n$ linearly independent eigenvectors of $A$, and let $P = [\, \vec{p}_1 \; \vec{p}_2 \; \cdots \; \vec{p}_n \,]$. Write $\lambda_i$ for the $i$th eigenvalue, so $A\vec{p}_i = \lambda_i \vec{p}_i$ for each $i$, and let $D$ be the diagonal matrix with the $\lambda_i$'s on the diagonal. Then

$$AP = A[\ \vec{p}_1\ \vec{p}_2\ \cdots\ \vec{p}_n\ ] = [\ \lambda_1\vec{p}_1\ \ \lambda_2\vec{p}_2\ \ \cdots\ \lambda_n\vec{p}_n\ ]$$
$$= [\ \vec{p}_1\ \vec{p}_2\ \cdots\ \vec{p}_n\ ]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

so $P^{-1}AP = D$, as required.

On the other hand, if $P^{-1}AP = D$ and $D$ is diagonal, then $AP = PD$, and if follows from an argument like the one above that the columns of $P$ are eigenvectors of $A$, and the eigenvalues are the diagonal entries of $D$. $\square$

This theorem is one of the main reasons we want to be able to find eigenvectors of a matrix. Moreover, the more eigenvectors the better, so this motivates allowing complex eigenvectors. We're going to say a lot more about diagonalization.