Math 1600A Lecture 3, Section 002

Announcements:

More texts, solutions manuals and packages coming soon.

Read Section 1.3 for next class. Work through recommended homework questions.
Here are scans of the questions from sections 1.1 and 1.2.

Tutorials start September 18, and include a quiz covering until Monday's lecture.
More details on Monday.

Bonus office hours: today, 2:30-3:30, MC103B.
Also, if you can't make it to my office hours, feel free to attend Hugo Bacard's office hours.

Help Centers Monday-Friday 2:30-6:30 in MC 106 starting Sept 19.

Questions after class:
Questions about homework problems should be asked at the Help Centers, tutorials or office hours.
Questions about my lecture should be asked **during class**, as others are likely to have the same question. (And it makes me feel like you are paying attention. :-)
Questions after class should be limited to questions specific to you. There are too many students for me to answer general questions.

Lecture notes (this page) available from course web page by clicking on our class times.
Answers to lots of administrative questions are available on the course web page as well.

Review of last lecture:

Many properties that hold for real numbers also hold for vectors in $\mathbb{R}^n$: Theorem 1.1. But we'll see differences later.

Definition: A vector $\vec{v}$ is a **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there exist scalars $c_1, c_2, \dots, c_k$ (called coefficients) such that

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k.$$

We also call the coefficients **coordinates** when we are thinking of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ as defining a new coordinate system.

Vectors modulo m:

$\mathbb{Z}_m = \{0, 1, \dots, m-1\}$ with addition and multiplication taken modulo m. That means that the answer is the remainder after division by m.

For example, in $\mathbb{Z}_{10}$, $\quad 8 \cdot 8 = 64 = 4 \pmod{10}$.

$\mathbb{Z}_m^n$ is the set of vectors with n components, each of which is in $\mathbb{Z}_m$.

New material

To find solutions to an equation such as

$$6x = 6 \pmod{8}$$

you can simply try all possible values of x. In this case, 1 and 5 both work, and no other value works.

Note that you can not in general **divide** in $\mathbb{Z}_m$, only add, subtract and multiply.

Section 1.2: Length and Angle: The Dot Product

Definition: The **dot product** of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is the real number defined by

$$\vec{u} \cdot \vec{v} := u_1v_1 + \cdots + u_nv_n.$$

Since $\vec{u} \cdot \vec{v}$ is a *scalar*, the dot product is sometimes called the **scalar product**.

The dot product will be used to define length, distance and angles in $\mathbb{R}^n$.

Examples on whiteboard, including in $\mathbb{Z}_m$.

Theorem 1.2: For vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$ and c in $\mathbb{R}$:
(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
(c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$
(d) $\vec{u} \cdot \vec{u} \geq 0$
(e) $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

Again, very similar to how multiplication and addition of numbers works.

Explain (b) and (d) on whiteboard. (a) and (c) are explained in text.

Length from dot product

Definition: The **length** or **norm** of $\vec{v}$ is the scalar $\|\vec{v}\|$ defined by

$$\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Whiteboard: Pythagorean theorem in $\mathbb{R}^2$. Example in $\mathbb{R}^4$. $\|c\vec{v}\| = |c|\|\vec{v}\|$.

Definition: A vector of length 1 is called a **unit** vector.

Whiteboard: Unit vectors in $\mathbb{R}^2$. Unit vector in same direction as $\vec{v}$, formula and example. $\vec{e}_1, \vec{e}_2, \vec{e}_3$ in $\mathbb{R}^3$. **Standard unit vectors** in $\mathbb{R}^n$.
Picture for triangle inequality.

Theorem 1.5: The Triangle Inequality: For all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$,

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

Whiteboard: Example in $\mathbb{R}^2$: $[1, 0]$ and $[3, 4]$.

Distance from length

Thinking of vectors $\vec{u}$ and $\vec{v}$ as starting from the origin, we define the **distance** between them by the formula

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2},$$

generalizing the formula for the distance between points in the plane.

Example: The distance between $\vec{u} = [10, 10, 10, 10]$ and $\vec{v} = [11, 11, 11, 11]$ is

$$\sqrt{(-1)^2 + (-1)^2 + (-1)^2 + (-1)^2} = \sqrt{4} = 2.$$

Angles from dot product

The unit vector in $\mathbb{R}^2$ at angle θ from the x-axis is $\vec{u} = [\cos\theta, \sin\theta]$. Notice that

$$\vec{u} \cdot \vec{e}_1 = [\cos\theta, \sin\theta] \cdot [1, 0] = 1 \cdot \cos\theta + 0 \cdot \sin\theta = \cos\theta.$$

More generally, given vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$, one can show using the law of cosines that

$$\vec{u} \cdot \vec{v} = \cos\theta \, \|\vec{u}\| \, \|\vec{v}\|,$$

where θ is the angle between them.

In particular, $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$, since $|\cos\theta| \leq 1$.

This holds in $\mathbb{R}^n$ as well:

Theorem 1.4: The Cauchy-Schwarz Inequality: For all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$,

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|.$$

We can therefore use the dot product to *define* the **angle** between two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ by the formula

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}, \quad \text{i.e.,} \quad \theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}\right),$$

where we choose $0 \leq \theta \leq 180^\circ$. This makes sense because the RHS is between -1 and 1.

Section 1.2 continued in lecture 4.