

## Math 1600A Lecture 30, Section 2, 20 Nov 2013

### Announcements:

Today we finish 4.4. **Read** Markov chains part of Section 4.6 for Friday. Not covering Section 4.5, or rest of 4.6 (which contains many interesting applications!) Work through recommended [homework questions](#). **Some updated!**

**Tutorials:** Quiz 5 is this week. It covers Appendix C, 4.1, 4.2 and 4.3

**Office hour:** Wednesday, 12:30-1:30, MC103B.

**Help Centers:** Monday-Friday 2:30-6:30 in MC 106.

The **final exam** will take place on Mon Dec 9, 2-5pm.

See course home page for rooms and for the [conflict policy](#).

You should **already** have notified the registrar or your Dean (and me) of any conflicts!

**True/false:** Every polynomial of degree  $n$  has exactly  $n$  distinct roots over  $\mathbb{C}$ .

False. For example,  $(x - 1)^2$  has only the root 1. A polynomial of degree  $n$  has exactly  $n$  complex roots if you count them with multiplicity. Over  $\mathbb{R}$ , the sum of the multiplicities is at most  $n$ .

**True/false:** The complex eigenvalues of a matrix always come in conjugate pairs.

False. This is true if the matrix has only real entries, but

$$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

has  $i$  as an eigenvalue, but not  $\bar{i}$ .

**True/false:** If  $\lambda$  is an eigenvalue of  $A$  and  $k \geq 0$ , then  $\lambda^k$  is an eigenvalue of  $A^k$ .

True, since if  $A\vec{x} = \lambda\vec{x}$ , then  $A^k\vec{x} = A^{k-1}\lambda\vec{x} = \dots = \lambda^k\vec{x}$ .

**True/false:** The identity matrix is similar to every matrix.

False. Since  $PIP^{-1} = I$  for any invertible  $P$ , the identity matrix is only similar to itself.

**True/False:** If  $A$  and  $B$  have the same eigenvalues, then  $A$  and  $B$  are similar.

False. For example,  $I$  and  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  have the same eigenvalues, but aren't similar.

### Review of Section 4.4

**Definition:** Let  $A$  and  $B$  be  $n \times n$  matrices. We say that  $A$  is **similar** to  $B$  if there is an

invertible matrix  $P$  such that  $P^{-1}AP = B$ . When this is the case, we write  $A \sim B$ .

It is equivalent to say that  $AP = PB$  or  $A = PBP^{-1}$ .

It is tricky in general to find such a  $P$  when it exists. We'll learn a method that works in a certain situation in this section.

**Theorem 4.21:** Let  $A$ ,  $B$  and  $C$  be  $n \times n$  matrices. Then:

- $A \sim A$ .
- If  $A \sim B$  then  $B \sim A$ .
- If  $A \sim B$  and  $B \sim C$ , then  $A \sim C$ .

Similar matrices have a lot of properties in common.

**Theorem 4.22:** Let  $A$  and  $B$  be similar matrices. Then:

- $\det A = \det B$
- $A$  is invertible iff  $B$  is invertible.
- $A$  and  $B$  have the same rank.
- $A$  and  $B$  have the same characteristic polynomial.
- $A$  and  $B$  have the same eigenvalues.

**Question:** Are  $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$  similar?

## Diagonalization

**Definition:**  $A$  is **diagonalizable** if it is similar to some diagonal matrix.

If  $A$  is similar to a diagonal matrix  $D$ , then  $D$  must have the eigenvalues of  $A$  on the diagonal. But how to find  $P$ ?

**Theorem 4.23:** Let  $A$  be an  $n \times n$  matrix. Then  $A$  is diagonalizable if and only if  $A$  has  $n$  linearly independent eigenvectors.

More precisely, there exist an invertible matrix  $P$  and a diagonal matrix  $D$  with  $P^{-1}AP = D$  if and only if the columns of  $P$  are  $n$  linearly independent eigenvectors of  $A$  and the diagonal entries of  $D$  are the corresponding eigenvalues in the same order.

This theorem is one of the main reasons we want to be able to find eigenvectors of a matrix. Moreover, the more eigenvectors the better, so this motivates allowing complex eigenvectors.

In section 4.3, we saw:

**Theorem 4.20:** If  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  are eigenvectors of  $A$  corresponding to **distinct**

eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_m$ , then  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  are linearly independent.

### New material: Section 4.4 continued

**Example:** Is the matrix  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$  diagonalizable?

Yes. The eigenvalues are 1, 4 and 6, and for each one there is at least one eigenvector. These are linearly independent (by Theorem 4.20), and there are three of them, so  $A$  is diagonalizable (by Theorem 4.23).

To **find** the matrix  $P$  explicitly, we need to solve the three systems to find the eigenvectors.

**Theorem 4.25:** If  $A$  in an  $n \times n$  matrix with  $n$  distinct eigenvalues, then  $A$  is diagonalizable.

**Example 4.25:** Is  $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$  diagonalizable? If so, find a matrix  $P$  that diagonalizes it.

**Solution:** In [Example 4.18](#) we found that the eigenvalues are  $\lambda = 1$  (with algebraic multiplicity 2) and  $\lambda = 2$  (with algebraic multiplicity 1). A basis for  $E_1$  is  $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$  and a

basis for  $E_2$  is  $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ . Since every eigenvector is a scalar multiple of one of these, it is not possible to find three linearly independent eigenvectors. So  $A$  is not diagonalizable.

**Example 4.26:** Is  $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$  diagonalizable? If so, find a matrix  $P$  that diagonalizes it.

**Solution:** In [Example 4.19](#) (done mostly on whiteboard, but also in text) we found that the eigenvalues are  $\lambda = 0$  (with algebraic multiplicity 2) and  $\lambda = -2$  (with algebraic multiplicity 1). A basis for  $E_0$  is  $\vec{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  and  $\vec{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ . A basis for  $E_{-2}$  is

$\vec{p}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$ . These are linearly independent (see below). Thus

$$P = [\vec{p}_1 \ \vec{p}_2 \ \vec{p}_3] = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

is invertible, and by the theorem, we must have

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D$$

(Note that to check an answer like this, it is usually easiest to check that  $AP = PD$ . Do so!)

Note: Different orders work too.

**Theorem 4.24:** If  $\lambda_1, \dots, \lambda_k$  are distinct eigenvalues of  $A$  and, for each  $i$ ,  $\mathcal{B}_i$  is a basis for the eigenspace  $E_{\lambda_i}$ , then the union of the  $\mathcal{B}_i$ 's is a linearly independent set.

The proof of this is similar to the proof of Theorem 4.20, where we had only one non-zero vector in each eigenspace.

Combining Theorems 4.23 and 4.24 gives the following important consequence:

**Theorem:** An  $n \times n$  matrix is diagonalizable if and only if the sum of the geometric multiplicities of the eigenvalues is  $n$ .

Look at Examples 4.25 and 4.26 again.

So it is important to understand the geometric multiplicities better. Here is a helpful result:

**Lemma 4.26:** If  $\lambda_1$  is an eigenvalue of an  $n \times n$  matrix  $A$ , then

$$\text{geometric multiplicity of } \lambda_1 \leq \text{algebraic multiplicity of } \lambda_1$$

We'll prove this in a minute. First, let's look at what it implies:

Let  $A$  be an  $n \times n$  matrix with distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ . Let their geometric multiplicities be  $g_1, g_2, \dots, g_k$  and their algebraic multiplicities be  $a_1, a_2, \dots, a_k$ . We know

$$g_i \leq a_i \quad \text{for each } i$$

and so

$$g_1 + \cdots + g_k \leq a_1 + \cdots + a_k \leq n$$

So the only way to have  $g_1 + \cdots + g_k = n$  is to have  $g_i = a_i$  for each  $i$  and  $a_1 + \cdots + a_k = n$ .

This gives the **main theorem** of the section:

**Theorem 4.27 (The Diagonalization Theorem):** Let  $A$  be an  $n \times n$  matrix with distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ . Let their geometric multiplicities be  $g_1, g_2, \dots, g_k$  and their algebraic multiplicities be  $a_1, a_2, \dots, a_k$ . Then the following are equivalent:

- $A$  is diagonalizable.
- $g_1 + \cdots + g_k = n$ .
- $g_i = a_i$  for each  $i$  and  $a_1 + \cdots + a_k = n$ .

**Note:** This is stated incorrectly in the text. The red part must be added unless you are working over  $\mathbb{C}$ , in which case it is automatic that  $a_1 + \cdots + a_k = n$ . With the way I have stated it, it is correct over  $\mathbb{R}$  or over  $\mathbb{C}$ .

**Example:** Is  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  diagonalizable?

It **depends**. If we are working over  $\mathbb{R}$ , there are no eigenvalues and no eigenvectors, so no, it is not diagonalizable, and (a), (b) and (c) all fail.

If we are working over  $\mathbb{C}$ , then  $i$  and  $-i$  are eigenvalues, and are distinct, so  $A$  is diagonalizable, and (b) and (c) hold too. Note that in this case,  $P$  will be complex: To find  $P$ , we first find that corresponding eigenvectors are  $\vec{p}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$  and  $\vec{p}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ . So if we take

$$P = [\vec{p}_1 \ \vec{p}_2] = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}$$

we find that

$$P^{-1}AP = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = D$$

We still need to prove Lemma 4.26, and will do so next class.