Math 1600A Lecture 36, Section 2, 4 Dec 2013

Announcements:

Today we finish Section 5.4 and finish the course material. **Read** the whole book for Monday. Work through recommended homework questions **and more**.

Tutorials: This week: review. Bring questions.
Office hour: Wednesday, 12:30-1:30, MC103B.
Help Centers: Monday-Friday 2:30-6:30 in MC 106.
Review Session: Friday in class; bring questions. Also, Sunday, Dec 8, 10am-11am, in MC110.

Final exam: Covers whole course, with an emphasis on the material in Chapters 4 and 5 (after the second midterm). It does *not* cover $\mathbb{Z}_m$, code vectors, Markov chains or network analysis. Everything else we covered in class is considered exam material. Questions are similar to textbook questions, midterm questions and quiz questions.

Review of Section 5.4 from last class

Section 5.4: Orthogonal Diagonalization of Symmetric Matrices

In Section 4.4 we learned all about diagonalizing a square matrix A. One of the difficulties that arose is that a matrix with real entries can have complex eigenvalues. In this section, we focus on the case where A is a symmetric matrix, and we will show that the eigenvalues of A are always real and that A is always diagonalizable!

Symmetric matrices are important in applications. For example, in quantum theory, they correspond to observable quantities.

Recall that a square matrix A is **symmetric** if $A^T = A$.

Examples: $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

Non-examples: $\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 5 \\ 3 & 2 & 6 \end{bmatrix}$.

New material

Example 5.16: If possible, diagonalize $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$. On whiteboard.

Definition: A square matrix A is **orthogonally diagonalizable** if there exists an orthogonal matrix Q such that $Q^T AQ$ is a diagonal matrix D.

Notice that if A is orthogonally diagonalizable, then $Q^T AQ = D$, so $A = QDQ^T$. Therefore

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A.$$

We have proven:

Theorem 5.17: If A is orthogonally diagonalizable, then A is symmetric.

The rest of this section is working towards proving that every symmetric matrix A is orthogonally diagonalizable. I'll organize this a bit more efficiently than the textbook.

Theorem 5.19: If A is a symmetric matrix, then eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

In non-symmetric examples we've seen earlier, the eigenvectors were not orthogonal.

Proof: Suppose $\vec{v}_1$ and $\vec{v}_2$ are eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2. Then we have

$$\begin{aligned} \lambda_1(\vec{v}_1 \cdot \vec{v}_2) &= (\lambda_1 \vec{v}_1) \cdot \vec{v}_2 = (A\vec{v}_1) \cdot \vec{v}_2 = (A\vec{v}_1)^T \vec{v}_2 \\ &= \vec{v}_1^T A^T \vec{v}_2 = \vec{v}_1^T (A\vec{v}_2) = \vec{v}_1^T \lambda_2 \vec{v}_2 = \lambda_2(\vec{v}_1 \cdot \vec{v}_2) \end{aligned}$$

So $(\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0$, which implies that $\vec{v}_1 \cdot \vec{v}_2 = 0$. $\square$

Theorem 5.18: If A is a real symmetric matrix, then the eigenvalues of A are real.

To prove this, we have to recall some facts about complex numbers. If $z = a + bi$, then its complex conjugate is $\bar{z} = a - bi$, which is the reflection in the real axis. So z is real if and only if $z = \bar{z}$.

Proof: Suppose that λ is an eigenvalue of A with eigenvector $\mathbf{v}$. Then the complex conjugate $\bar{\mathbf{v}}$ is an eigenvector with eigenvalue $\bar{\lambda}$, since

$$A\bar{\mathbf{v}} = \bar{A}\bar{\mathbf{v}} = \overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}.$$

If $\lambda \neq \bar{\lambda}$, then Theorem 5.19 shows that $\mathbf{v} \cdot \bar{\mathbf{v}} = 0$.

But if $\mathbf{v} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ then $\bar{\mathbf{v}} = \begin{bmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{bmatrix}$ and so

$$\mathbf{v} \cdot \bar{\mathbf{v}} = z_1 \bar{z}_1 + \cdots + z_n \bar{z}_n = |z_1|^2 + \cdots + |z_n|^2 \neq 0$$

since $\mathbf{v} \neq \vec{0}$. Therefore, $\lambda = \bar{\lambda}$, so λ is real. $\square$

Example 5.17 and 5.18: The eigenvalues of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ are 4 and 1, with eigenspaces

$$E_4 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) \quad \text{and} \quad E_1 = \text{span}(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix})$$

We see that every vector in E_1 is orthogonal to every vector in E_4. (In fact, $E_1 = E_4^{\perp}$.)

But notice that the vectors in E_1 aren't necessarily orthogonal to each other. However, we can apply Gram-Schmidt to get an orthogonal basis for E_1:

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

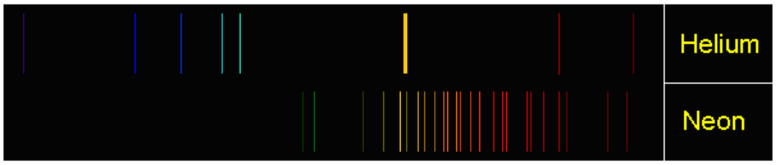
$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

We normalize the three basis eigenvectors and put them in the columns of a matrix $Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}$. Then $Q^T A Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so A is **orthogonally** diagonalizable.

The spectral theorem

The set of eigenvalues of a matrix are called its **spectrum** because the spectral lines you see when light from an atom is sent through a prism correspond to the eigenvalues of a certain matrix.



Theorem 5.20 (The spectral theorem): Let A be an $n \times n$ real matrix. Then A is symmetric if and only if A is orthogonally diagonalizable.

Proof: We have seen that every orthogonally diagonalizable matrix is symmetric.

We also know that if A is symmetric, then it's eigenvectors for distinct eigenvalues are orthogonal. So, by using Gram-Schmidt on the eigenvectors with the same eigenvalue, we get an orthogonal set of eigenvectors.

The only thing that isn't clear is that we get n eigenvectors. The argument here is a bit complicated. See the text. $\square$.

Method for orthogonally diagonalizing a real symmetric $n \times n$ matrix A:
1. Find all eigenvalues. They will all be real, and the algebraic multiplicities will add up to n.
2. Find a basis for each eigenspace.
3. If an eigenspace has dimension greater than one, use Gram-Schmidt to create an orthogonal basis of that eigenspace.
4. Normalize all basis vectors. Put them in the columns of Q, and make the eigenvalues (in the same order) the diagonal entries of a diagonal matrix D.
5. Then $Q^T A Q = D$.

Note that A can be expressed in terms of its eigenvectors $\vec{q}_1, \ldots, \vec{q}_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ (repeated according to their multiplicity) as

$$\begin{aligned} A = QDQ^T &= [\, \vec{q}_1 \cdots \vec{q}_n \,] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{q}_1^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{bmatrix} \\ &= [\, \lambda_1 \vec{q}_1 \cdots \lambda_n \vec{q}_n \,] \begin{bmatrix} \vec{q}_1^{\,T} \\ \vdots \\ \vec{q}_n^{\,T} \end{bmatrix} \\ &= \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \cdots + \lambda_n \vec{q}_n \vec{q}_n^{\,T} \end{aligned}$$

This is called the **spectral decomposition** of A.

Note that the $n \times n$ matrix $\vec{q}_1 \vec{q}_1^{\,T}$ sends a vector $\vec{x}$ to

$\vec{q}_1\,\vec{q}_1^{\,T}\,\vec{x} = (\vec{q}_1 \cdot \vec{x})\,\vec{q}_1 = \text{proj}_{\vec{q}_1}(\vec{x})$, so it is orthogonal projection onto $\text{span}(\vec{q}_1)$. Thus you can compute $A\vec{x}$ by projecting $\vec{x}$ onto each $\vec{q}_i$, multiplying by λ_i, and adding the results.

Example 5.20: Find a 2×2 matrix with eigenvalues 3 and -2 and corresponding eigenvectors $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

Method 1: Let $P = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. Then

$$\begin{aligned} A = PDP^{-1} &= \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 9 & 8 \\ 12 & -6 \end{bmatrix} \frac{1}{25} \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{25} \begin{bmatrix} -5 & 60 \\ 60 & -30 \end{bmatrix} = \begin{bmatrix} -1/5 & 12/5 \\ 12/5 & -6/5 \end{bmatrix} \end{aligned}$$

This didn't use anything from this section and works for any diagonalizable matrix.

Method 2: First normalize the eigenvectors to have length 1. Then use the spectral decomposition:

$$\begin{aligned} A &= \lambda_1\,\vec{q}_1\,\vec{q}_1^{\,T} + \lambda_2\,\vec{q}_2\,\vec{q}_2^{\,T} \\ &= 3\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 \end{bmatrix} - 2\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} \begin{bmatrix} -4/5 & 3/5 \end{bmatrix} \\ &= 3\begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix} - 2\begin{bmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{bmatrix} = \begin{bmatrix} -1/5 & 12/5 \\ 12/5 & -6/5 \end{bmatrix} \end{aligned}$$

This method only works because the given vectors are orthogonal.

See Example 5.19 in the text for another example.