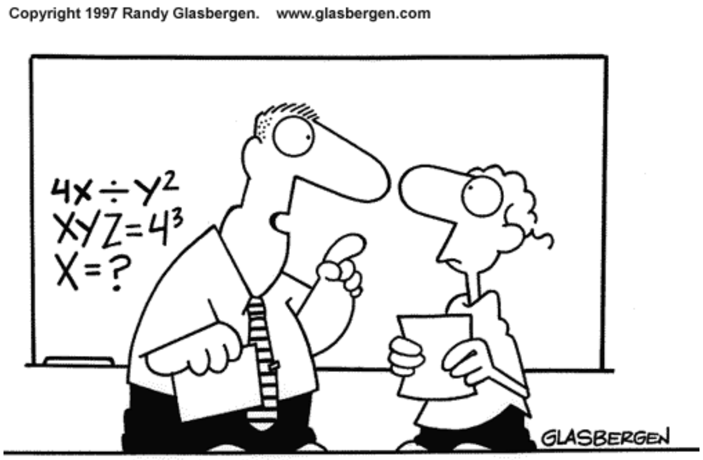
Math 1600A Lecture 4, Section 002



"Algebra class will be important to you later in life because there's going to be a test six weeks from now."

(Actually, a quiz this week and a midterm in 2 1/2 weeks...)

Announcements:

More texts, solutions manuals and packages coming soon.

Continue reading Section 1.3 for next class. Work through recommended homework questions. Here are scans of the questions from sections 1.1 and 1.2. Scans of the rest of Sections 1.1 and 1.2 coming today.

Tutorials start **this week**, and include a **quiz** covering until the end of Section 1.2. It does not cover Section 1.3 or the Exploration after Section 1.2.
The quizzes last 20 minutes, and are at the end of the tutorial, so you have time for questions at the beginning.
Questions *similar* to homework questions, but may be slightly different. Possibly some true/false questions.
You must write in the tutorial you are registered in.
Different sections have different quizzes, but it is still considered an academic offense to share information about quizzes.

Office hour: today, 1:30-2:30, MC103B.
Also, if you can't make it to my office hours, feel free to attend Hugo Bacard's

office hours.

Help Centers Monday-Friday 2:30-6:30 in MC 106 starting Thursday.

Lecture notes (this page) available from course web page by clicking on our class times.

Partial review of last lecture:

Section 1.2: Length and Angle: The Dot Product

Definition: The **dot product** or **scalar product** of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is the real number defined by

$$\vec{u} \cdot \vec{v} := u_1 v_1 + \cdots + u_n v_n.$$

This has nice properties; see Theorem 1.2.

Definition: The **length** or **norm** of $\vec{v}$ is the scalar $\|\vec{v}\|$ defined by

$$\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

A vector of length 1 is called a **unit** vector.

Theorem 1.5: The Triangle Inequality: For all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$,

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|.$$

We define the **distance** between vectors $\vec{u}$ and $\vec{v}$ by the formula

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}.$$

Angles from dot product

Theorem 1.4: The Cauchy-Schwarz Inequality: For all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$,

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|.$$

We can therefore use the dot product to *define* the **angle** between two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ by the formula

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\,, \quad \text{i.e.,} \quad \theta = \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right),$$

where we choose $0 \leq \theta \leq 180^\circ$. This makes sense because the RHS is between -1 and 1.

New material

The help remember the formula for $\cos\theta$, note that the denominator normalizes the two vectors to be unit vectors.

An applet illustrating the dot product. If it doesn't work, try the java version.

Whiteboard: Example: Angle between $\vec{u} = [1, 2, 1, 1, 1]$ and $\vec{v} = [0, 3, 0, 0, 0]$.

For a random example, you'll need a calculator, but for hand calculations you can remember these cosines:

$$\cos 0^\circ = \frac{\sqrt{4}}{2} = 1, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}\,, \quad \cos 45^\circ = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\,, \quad \cos 60^\circ = \frac{\sqrt{1}}{2} = \frac{1}{2}$$

using the usual triangles.

Orthogonal Vectors

How can we tell whether two vectors are orthogonal / perpendicular?
Easy: $\theta = 90^\circ$ is the only angle for which $\cos\theta = 0$.
So $\vec{u}$ and $\vec{v}$ are **orthogonal** if and only if $\vec{u}\cdot\vec{v} = 0$.

Whiteboard: Example: $\vec{u} = [1, 2, 3]$ and $\vec{v} = [1, 1, -1]$ in $\mathbb{R}^3$.
Also, $\vec{u} = [1, 2, 3]$ and $\vec{v} = [1, 1, 1]$ in $\mathbb{Z}_3^3$.

Pythagorean theorem in $\mathbb{R}^n$: If $\vec{u}$ and $\vec{v}$ are orthogonal, then

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2.$$

Whiteboard: Explain, using Theorem 1.2.

Projections

Use whiteboard to derive formula for **the projection of $\vec{v}$ onto $\vec{u}$**:

$$\operatorname{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}.$$

To help remember the formula, note that the denominator ensures that the answer does not depend on the length of $\vec{u}$.

The same applet is useful for understanding projections as well. Java version.

Questions

Suppose $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^n$ such that $\vec{u}\cdot\vec{v} = \vec{u}\cdot\vec{w}$ and $\vec{u} \neq \vec{0}$. Does it follow that $\vec{v} = \vec{w}$?

Suppose that $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$. Does it follow that it is orthogonal to $2\vec{v} + 3\vec{w}$?

Suppose I tell you that $\vec{u}\cdot\vec{v} = 1/2$ and $\vec{u}\cdot\vec{w} = -1$. What is $\vec{u}\cdot(2\vec{v} + 3\vec{w})$?

Does $\operatorname{proj}_{\vec{u}}(\vec{v})$ always point in the same direction as $\vec{u}$?

Answers on whiteboard.

Questions from the class, if time.