Math 1600 Lecture 14, Section 2, 6 Oct 2014

Announcements:

Continue **reading** Section 3.1 (partitioned matrices) and Section 3.2 for next class. Work through recommended homework questions.

Quiz 4 is this week, and will focus on the material in Section 2.4 (networks) and the parts of 3.1 and 3.2 we finish today.

Office hour: Today, 3:00-3:30, MC103B.

Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Partial review of Lecture 13:

Section 3.1: Matrix Operations

Definition: An $m \times n$ **matrix** A is a rectangular array of numbers called the **entries**, with m rows and n columns. A is called **square** if $m = n$.

The entry in the ith row and jth column of A is usually written a_{ij} or sometimes A_{ij}.

The **diagonal entries** are $a_{11}, a_{22}, \ldots$

If A is square and the nondiagonal entries are all zero, then A is called a **diagonal matrix**.

$$\begin{bmatrix} 1 & -3/2 & \pi \\ \sqrt{2} & 2.3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{not square or diagonal} \qquad \text{square} \qquad \text{diagonal} \qquad \text{diagonal}$$

Definition: A diagonal matrix with all diagonal entries equal is called a **scalar matrix**. A scalar matrix with diagonal entries all equal to 1 is an

identity matrix.

All of these are scalar matrices:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{identity matrix} \qquad \text{scalar} \qquad \text{zero matrix}$$

Note: Identity $\implies$ scalar $\implies$ diagonal $\implies$ square.

Matrix addition and scalar multiplication

Our first two operations are just like for vectors:

Definition: If A and B are both $m \times n$ matrices, then their **sum** $A + B$ is the $m \times n$ matrix obtained by adding the corresponding entries of A and B: $A + B = [a_{ij} + b_{ij}]$.

Definition: If A is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** cA is the $m \times n$ matrix obtained by multiplying each entry by c: $cA = [c\,a_{ij}]$.

New material: Section 3.2: Matrix Algebra

Addition and scalar multiplication for matrices behave **exactly** like addition and scalar multiplication for vectors, with the entries just written in a rectangle instead of in a row or column.

Theorem 3.2: Let A, B and C be matrices of the same size, and let c and d be scalars. Then:

(a) $A + B = B + A$ (comm.)
(b) $(A + B) + C = A + (B + C)$ (assoc.)
(c) $A + O = A$
(d) $A + (-A) = O$
(e) $c(A + B) = cA + cB$ (dist.)
(f) $(c + d)A = cA + dA$ (dist.)
(g) $c(dA) = (cd)A$
(h) $1A = A$

Compare to Theorem 1.1.

This means that all of the concepts for vectors transfer to matrices.

E.g., manipulating matrix equations:

$$2(A+B)-3(2B-A)=2A+2B-6B+3A=5A-4B.$$

We define a **linear combination** to be a matrix of the form:

$$c_1A_1+c_2A_2+\cdots+c_kA_k.$$

And we can define the **span** of a set of matrices to be the set of all their linear combinations.

And we can say that the matrices $A_1, A_2, \ldots, A_k$ are **linearly independent** if

$$c_1A_1+c_2A_2+\cdots+c_kA_k=O$$

has only the trivial solution $c_1=\cdots=c_k=0$, and are **linearly dependent** otherwise.

Our techniques for vectors also apply to answer questions such as:

Example 3.16 (a): Suppose

$$A_1=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\ A_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ A_3=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix},\ B=\begin{bmatrix}1 & 4\\ 2 & 1\end{bmatrix}$$

Is B a linear combination of A_1, A_2 and A_3?

That is, are there scalars c_1, c_2 and c_3 such that

$$c_1\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}+c_2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+c_3\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 4\\ 2 & 1\end{bmatrix}?$$

Rewriting the left-hand side gives

$$\begin{bmatrix}c_2+c_3 & c_1+c_3\\ -c_1+c_3 & c_2+c_3\end{bmatrix}=\begin{bmatrix}1 & 4\\ 2 & 1\end{bmatrix}$$

and this is equivalent to the system

$$\begin{aligned} c_2 + c_3 &= 1 \\ c_1 \qquad + c_3 &= 4 \\ -c_1 \qquad + c_3 &= 2 \\ c_2 + c_3 &= 1 \end{aligned}$$

and we can use row reduction to determine that there is a solution, and to find it if desired: $c_1 = 1, c_2 = -2, c_3 = 3$, so $A_1 - 2A_2 + 3A_3 = B$.

This works exactly as if we had written the matrices as column vectors and asked the same question.

See also Examples 3.16(b), 3.17 and 3.18 in text.

More review of Lecture 13:

Matrix multiplication

Definition: If A is $m \times n$ and B is $n \times r$, then the **product** $C = AB$ is the $m \times r$ matrix whose i, j entry is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

This is the dot product of the ith row of A with the jth column of B.

$$\begin{array}{cccc} A & B & = & AB \\ m \times n & n \times r & & m \times r \end{array}$$

Powers

In general, $A^2 = AA$ doesn't make sense. But if A is $n \times n$ (square), then it makes sense to define the **power**

$$A^k = AA \cdots A \quad \text{with } k \text{ factors.}$$

We write $A^1 = A$ and $A^0 = I_n$.

We will see in a moment that $(AB)C = A(BC)$, so the expression for A^k is unambiguous. And it follows that

$$A^r A^s = A^{r+s} \qquad \text{and} \qquad (A^r)^s = A^{rs}$$

for all nonnegative integers r and s.

New material: Section 3.2: Matrix Algebra (continued)

Properties of Matrix Multiplication and Powers

This is new ground, as you can't multiply vectors.

For the most part, matrix multiplication behaves like multiplication of real numbers, but there are several differences:

Example 3.13 on board: Powers of

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Question: Is there a nonzero matrix A such that $A^2 = O$?

Yes. For example, take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{or} \qquad A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}.$$

Challenge problems: (1) Find a 3×3 matrix A such that $A^2 \neq O$ but $A^3 = O$.

(2) Find a 2×2 matrix A such that $A \neq I_2$ but $A^3 = I_2$.

I'll come back to these next class.

Example on board: Tell me the entries of two 2×2 matrices A and B, and let's compute AB and BA.

So we've seen:

We can have $A \neq O$ but $A^k = O$ for some $k > 1$.
We can have $B \neq \pm I$, but $B^4 = I$.
We can have $AB \neq BA$.

These are good material for true/false questions...

But most expected properties **do** hold:

Theorem 3.3: Let A, B and C be matrices of the appropriate sizes, and let k be a scalar. Then:

(a) $A(BC) = (AB)C$	(associativity)
(b) $A(B + C) = AB + AC$	(left distributivity)
(c) $(A + B)C = AC + BC$	(right distributivity)
(d) $k(AB) = (kA)B = A(kB)$	(no cool name)
(e) $I_m A = A = A I_n$ if A is $m \times n$	(identity)

The text proves (b) and half of (e). (c) and the other half of (e) are the same, with right and left reversed.

Proof of (d):

$$\begin{aligned}(k(AB))_{ij} &= k(AB)_{ij} = k(\mathrm{row}_i(A) \cdot \mathrm{col}_j(B)) \\ &= (k\,\mathrm{row}_i(A)) \cdot \mathrm{col}_j(B) = \mathrm{row}_i(kA) \cdot \mathrm{col}_j(B) \\ &= ((kA)B)_{ij}\end{aligned}$$

so $k(AB) = (kA)B$. The other part of (d) is similar. $\square$

Proof of (a): (Using A_{ij} notation for matrix entries.)

$$((AB)C)_{ij} = \sum_k (AB)_{ik} C_{kj} = \sum_k \sum_l A_{il} B_{lk} C_{kj}$$
$$= \sum_l \sum_k A_{il} B_{lk} C_{kj} = \sum_l A_{il} (BC)_{lj} = (A(BC))_{ij}$$

so $A(BC) = (AB)C$. $\square$

Example on board: $AI = A$.

Example on board: Solve

$$2(X - A) + (A + B)(B + I) = 0$$

for X in terms of A and B.

Example 3.20: If A and B are square matrices of the same size, is $(A + B)^2 = A^2 + 2AB + B^2$?

Solution: Using Theorem 3.3, we find:

$$\begin{aligned}(A + B)^2 &= (A + B)(A + B) \\ &= (A + B)A + (A + B)B \\ &= A^2 + BA + AB + B^2.\end{aligned}$$

Suppose $A^2 + BA + AB + B^2 = A^2 + 2AB + B^2$. Subtracting $A^2 + AB + B^2$ from both sides gives $BA = AB$. So the answer is "No, unless A and B commute."

Note: Theorem 3.3 shows that a scalar matrix kI_n commutes with *every* $n \times n$ matrix A. So

$$(A + kI_n)^2 = A^2 + 2A(kI_n) + (kI_n)^2 = ?$$

(I_n is like the number 1.)

True/false: If A is 2×3 and B is 3×2, then we always have $AB \neq BA$.

True. AB is 2×2 and BA is 3×3, so they can't be equal.

True/false: Every 1×1 matrix is a scalar matrix.

True. The only entry in a 1×1 matrix is on the diagonal, and "all" of the diagonal entries are equal.

It follows that all 1×1 matrices commute with each other.

Note: The non-commutativity of matrices is directly related to **quantum mechanics**. Observables in quantum mechanics are described by matrices, and if the matrices don't commute, then you can't know both quantities at the same time!

Next class: more from Sections 3.1 and 3.2: Transpose, symmetric matrices, partitioned matrices.