Math 1600 Lecture 18, Section 2, 17 Oct 2014

Enable java.

Announcements:

Continue **reading** Section 3.5. We aren't covering 3.4. Work through recommended homework questions.

Five **practice midterms** have been posted on the course web page.

Next office hour: Monday, 3:00-3:30, MC103B.

Help Centers: Monday-Friday 2:30-6:30 in MC 106, but not during reading week.

After today, we are halfway done the course!

Partial review of Section 3.3, Lectures 16 and 17:

Definition: An **inverse** of an $n \times n$ matrix A is an $n \times n$ matrix A' such that

$$AA' = I \quad \text{and} \quad A'A = I.$$

If such an A' exists, we say that A is **invertible**.

Theorem 3.6: If A is an invertible matrix, then its inverse is unique.

We write A^{-1} for **the** inverse of A, when A is invertible.

Theorem 3.8: The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$. When this is the case,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We call $ad - bc$ the **determinant** of A, and write it $\det A$.

Properties of Invertible Matrices

Theorem 3.9: Assume A and B are invertible matrices of the same size. Then:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$
b. If c is a non-zero scalar, then cA is invertible and $(cA)^{-1} = \frac{1}{c} A^{-1}$
c. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$ (socks and shoes rule)
d. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$
e. A^n is invertible for all $n \geq 0$ and $(A^n)^{-1} = (A^{-1})^n$

Remark: There is no formula for $(A+B)^{-1}$. In fact, $A+B$ might not be invertible, even if A and B are.

The fundamental theorem of invertible matrices:

Very important! Will be used repeatedly, and expanded later.

Theorem 3.12: Let A be an $n \times n$ matrix. The following are equivalent:
a. A is invertible.
b. $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.
c. $A\vec{x} = \vec{0}$ has only the trivial (zero) solution.
d. The reduced row echelon form of A is I_n.

Theorem 3.13: Let A be a square matrix. If B is a square matrix such that either $AB = I$ or $BA = I$, then A is invertible and $B = A^{-1}$.

Gauss-Jordan method for computing the inverse

Theorem 3.14: Let A be a square matrix. If a sequence of row operations reduces A to I, then the **same** sequence of row operations transforms I into A^{-1}.

This gives a general purpose method for determining whether a matrix A is invertible, and finding the inverse:

1. Form the $n \times 2n$ matrix $[A \mid I\,]$.

2. Use row operations to get it into reduced row echelon form.

3. If a zero row appears in the left-hand portion, then A is not invertible.

4. Otherwise, A will turn into I, and the right hand portion is A^{-1}.

New material: Section 3.5: Subspaces, basis, dimension and rank

This section contains some of the most important concepts of the course.

Subspaces

A generalization of lines and planes through the origin.

Definition: A **subspace** of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that:
1. The zero vector $\vec{0}$ is in S.
2. S is **closed under addition**: If $\vec{u}$ and $\vec{v}$ are in S, then $\vec{u}+\vec{v}$ is in S.
3. S is **closed under scalar multiplication**: If $\vec{u}$ is in S and c is any scalar, then $c\vec{u}$ is in S.

Conditions (2) and (3) together are the same as saying that S is **closed under linear combinations**.

Example: $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. Also, $S = \{\vec{0}\}$ is a subspace of $\mathbb{R}^n$.

Example: A plane $\mathcal{P}$ through the origin in $\mathbb{R}^3$ is a subspace. Applet.

Here's an algebraic argument. Suppose $\vec{v}_1$ and $\vec{v}_2$ are direction vectors for $\mathcal{P}$, so $\mathcal{P} = \mathrm{span}(\vec{v}_1, \vec{v}_2)$.
(1) $\vec{0}$ is in $\mathcal{P}$, since $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$.
(2) If $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2$ and $\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2$, then

$$\begin{aligned}\vec{u}+\vec{v} &= (c_1\vec{v}_1 + c_2\vec{v}_2) + (d_1\vec{v}_1 + d_2\vec{v}_2)\\ &= (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2\end{aligned}$$

which is in $\operatorname{span}(\vec{v}_1, \vec{v}_2)$ as well.
(3) For any scalar c,

$$c\vec{u} = c(c_1\vec{v}_1 + c_2\vec{v}_2) = (cc_1)\vec{v}_1 + (cc_2)\vec{v}_2$$

which is also in $\operatorname{span}(\vec{v}_1, \vec{v}_2)$.

On the other hand, a plane **not** through the origin is not a subspace. It of course fails (1), but the other conditions fail as well, as shown in the applet.

As another example, a line through the origin in $\mathbb{R}^3$ is also a subspace.

The **same** method as used above proves:

Theorem 3.19: Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. Then $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$.

See text. We call $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_k)$ the **subspace spanned by** $\vec{v}_1, \ldots, \vec{v}_k$. This generalizes the idea of a line or a plane through the origin.

Example: Is the set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x = y + z$ a subspace of $\mathbb{R}^3$?

Here S is the set of all vectors of the form $\begin{bmatrix} y+z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

That is, $S = \operatorname{span}(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix})$, so it is a subspace.

Alternatively, one could check the properties:
(1) This holds with $y = z = 0$.
(2) Since $\begin{bmatrix} y_1 + z_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} y_2 + z_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 + z_1 + y_2 + z_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$ is of the right form, this condition holds.

(3) Since $c \begin{bmatrix} y+z \\ y \\ z \end{bmatrix} = \begin{bmatrix} cy+cz \\ cy \\ cz \end{bmatrix}$, this condition holds too.

This is geometrically a plane through the origin, so our previous discussion applies as well.

See Example 3.38 in the text for a similar question.

Example: Is the set of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $x = y + z + 1$ a subspace of $\mathbb{R}^3$?

No, because it doesn't contain the zero vector. (The other properties don't hold either.)

Example: Is the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $y = \sin(x)$ a subspace of $\mathbb{R}^2$?

It does contain the zero vector. Let's check condition (3): Consider a vector $\begin{bmatrix} x \\ \sin(x) \end{bmatrix}$ in this set, and let c be a scalar. Then

$$c \begin{bmatrix} x \\ \sin(x) \end{bmatrix} = \begin{bmatrix} cx \\ c\sin(x) \end{bmatrix}$$

and $c\sin(x)$ is not usually equal to $\sin(cx)$.
To show that this is false, we give an explicit counterexample:
$\begin{bmatrix} \pi/2 \\ 1 \end{bmatrix}$ is in the set, but $2 \begin{bmatrix} \pi/2 \\ 1 \end{bmatrix} = \begin{bmatrix} \pi \\ 2 \end{bmatrix}$ is not in the set, since $\sin(\pi) = 0 \neq 2$.

Property (2) doesn't hold either.

Subspaces associated with matrices

Theorem 3.21: Let A be an $m \times n$ matrix and let N be the set of solutions of the homogeneous system $A\vec{x} = \vec{0}$. Then N is a subspace of $\mathbb{R}^n$.

Proof: (1) Since $A\,\vec{0}_n = \vec{0}_m$, the zero vector $\vec{0}_n$ is in N.
(2) Let $\vec{u}$ and $\vec{v}$ be in N, so $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$. Then

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$$

so $\vec{u} + \vec{v}$ is in N.
(3) If c is a scalar and $\vec{u}$ is in N, then

$$A(c\vec{u}) = cA\vec{u} = c\,\vec{0} = \vec{0}$$

so $c\vec{u}$ is in N. $\square$

Aside: At this point, the book states **Theorem 3.22**, which says that every linear system has no solution, one solution or infinitely many solutions, and gives a proof of this. We already know this is true, using Theorem 2.2 from Section 2.2 (see Lecture 9). The proof given here is in a sense better, since it doesn't rely on knowing anything about row echelon form, but I won't use class time to cover it.

Spans and null spaces are the *two main* sources of subspaces.

Definition: Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace $\mathrm{row}(A)$ of $\mathbb{R}^n$ spanned by the rows of A.
2. The **column space** of A is the subspace $\mathrm{col}(A)$ of $\mathbb{R}^m$ spanned by the columns of A.
3. The **null space** of A is the subspace $\mathrm{null}(A)$ of $\mathbb{R}^n$ consisting of the solutions to the system $A\vec{x} = \vec{0}$.

Example: The column space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\mathrm{span}(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix})$. A vector $\vec{b}$ is a linear combination of these columns if and only if the system $A\vec{x} = \vec{b}$ has a solution. But since A is invertible (its determinant is

$4-6=-2\neq 0$), every such system has a (unique) solution. So $\operatorname{col}(A)=\mathbb{R}^2$.

The row space of A is the same as the column space of A^T, so by a similar argument, this is all of $\mathbb{R}^2$ as well.

The null space of A consists of the vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $A\begin{bmatrix} x \\ y \end{bmatrix}=\vec{0}$. That is,

$$x\begin{bmatrix} 1 \\ 3 \end{bmatrix}+y\begin{bmatrix} 2 \\ 4 \end{bmatrix}=\vec{0}.$$

Since those columns are linearly independent, $\operatorname{null}(A)=\{\vec{0}\}$.

Example: The column space of $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is the span of the two columns, which is a subspace of $\mathbb{R}^3$. Since the columns are linearly independent, this is a plane through the origin in $\mathbb{R}^3$.

Determine whether $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ are in $\operatorname{col}(A)$. (On board.)

The row space of A is the span of the three rows. But we already saw that the span of the first two rows is $\mathbb{R}^2$, so the span of all three rows is also $\mathbb{R}^2$. So $\operatorname{row}(A)=\mathbb{R}^2$.

Again, since the columns are linearly independent, $\operatorname{null}(A)=\{\vec{0}\}$.

Example: Find the null space of $A=\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$.

We want to solve the system $A\vec{x} = \vec{0}$, so we row reduce $\left[\begin{array}{rr|r} 1 & 2 & 0 \\ -2 & -4 & 0 \end{array}\right]$ to $\left[\begin{array}{rr|r} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$. Then $y = t$ and $x = -2t$ are the solutions, so

$$\text{null}(A) = \left\{ \begin{bmatrix} -2t \\ t \end{bmatrix} \right\}.$$

Next we will explain the best way to describe a subspace.

Basis

We know that to describe a plane $\mathcal{P}$ through the origin, we can give two direction vectors $\vec{u}$ and $\vec{v}$ which are linearly independent. Then $\mathcal{P} = \text{span}(\vec{u}, \vec{v})$. We know that two vectors is always enough, and one vector will not work.

Definition: A **basis** for a subspace S of $\mathbb{R}^n$ is a set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ such that:
1. $S = \text{span}(\vec{v}_1, \ldots, \vec{v}_k)$, and
2. $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent.

Condition (2) ensures that none of the vectors is redundant, so we aren't being wasteful. Giving a basis for a subspace is a good way to "describe" it.

Example 3.42: The standard unit vectors $\vec{e}_1, \ldots, \vec{e}_n$ in $\mathbb{R}^n$ are linearly independent and span $\mathbb{R}^n$, so they form a basis of $\mathbb{R}^n$ called the **standard basis**.

Example: We saw above that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ span $\mathbb{R}^2$. They are also linearly independent, so they are a basis for $\mathbb{R}^2$.

Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are another basis for $\mathbb{R}^2$. A subspace will in general have many bases, but we'll see soon that they all have the same number of vectors! (Grammar: one basis, two bases.)

Next class we will continue talking about bases and will discuss systematic methods for finding the three subspaces associated to a matrix A.