Math 1600 Lecture 19, Section 2, 20 Oct 2014

Announcements:

Continue **reading** Section 3.5. Work through recommended homework questions.

Tutorials: Midterm review this week. No quiz.

Office hour: today, 3:00-3:30, MC103B.

Help Centers: Monday-Friday 2:30-6:30 in MC106.

Extra Midterm Review: Friday, October 24, details TBA. Bring questions.

Midterm: Saturday, October 25, 7-10pm. Rooms: A-E: UCC37. F-Ma: UCC56 (this room). Mc-Z: UCC146. It will cover the material up to and including today's lecture. **Practice midterms** are on website.

Partial review of Lecture 18:

Subspaces

Definition: A **subspace** of $\mathbb{R}^n$ is any collection S of vectors in $\mathbb{R}^n$ such that:
1. The zero vector $\vec{0}$ is in S.
2. S is **closed under addition**: If $\vec{u}$ and $\vec{v}$ are in S, then $\vec{u} + \vec{v}$ is in S.
3. S is **closed under scalar multiplication**: If $\vec{u}$ is in S and c is any scalar, then $c\vec{u}$ is in S.

Conditions (2) and (3) together are the same as saying that S is **closed under linear combinations**.

Example: $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. Also, $S = \{\vec{0}\}$ is a subspace of $\mathbb{R}^n$.

A line or plane through the origin in $\mathbb{R}^3$ is a subspace. Applet.

On the other hand, a plane **not** through the origin is not a subspace. It of course fails (1), but the other conditions fail as well, as shown in the applet.

Theorem 3.19: Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. Then $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$.

Subspaces associated with matrices

Theorem 3.21: Let A be an $m \times n$ matrix and let N be the set of solutions of the homogeneous system $A\vec{x} = \vec{0}$. Then N is a subspace of $\mathbb{R}^n$.

Spans and null spaces are the *two main* sources of subspaces.

Definition: Let A be an $m \times n$ matrix.

1. The **row space** of A is the subspace $\operatorname{row}(A)$ of $\mathbb{R}^n$ spanned by the rows of A.
2. The **column space** of A is the subspace $\operatorname{col}(A)$ of $\mathbb{R}^m$ spanned by the columns of A.
3. The **null space** of A is the subspace $\operatorname{null}(A)$ of $\mathbb{R}^n$ consisting of the solutions to the system $A\vec{x} = \vec{0}$.

Example: The column space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is $\operatorname{span}(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix})$, which we saw is all of $\mathbb{R}^2$. We also saw that the row space of A is $\mathbb{R}^2$ and the null space is $\{\vec{0}\}$.

Example: The column space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is the span of the two columns, which is a subspace of $\mathbb{R}^3$. Since the columns are linearly independent, this is a plane through the origin in $\mathbb{R}^3$.

Basis

We know that to describe a plane $\mathcal{P}$ through the origin, we can give two direction vectors $\vec{u}$ and $\vec{v}$ which are linearly independent. Then $\mathcal{P} = \text{span}(\vec{u}, \vec{v})$. We know that two vectors is always enough, and one vector will not work.

Definition: A **basis** for a subspace S of $\mathbb{R}^n$ is a set of vectors $\vec{v}_1, \ldots, \vec{v}_k$ such that:
1. $S = \text{span}(\vec{v}_1, \ldots, \vec{v}_k)$, and
2. $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent.

Condition (2) ensures that none of the vectors is redundant, so we aren't being wasteful. Giving a basis for a subspace is a good way to "describe" it.

Example 3.42: The standard unit vectors $\vec{e}_1, \ldots, \vec{e}_n$ in $\mathbb{R}^n$ are linearly independent and span $\mathbb{R}^n$, so they form a basis of $\mathbb{R}^n$ called the **standard basis**.

Example: We saw above that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ span $\mathbb{R}^2$. They are also linearly independent, so they are a basis for $\mathbb{R}^2$.

Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are another basis for $\mathbb{R}^2$. A subspace will in general have many bases, but we'll see soon that they all have the same number of vectors! (Grammar: one basis, two bases.)

New material

Example: Let $\mathcal{P}$ be the plane through the origin with direction vectors $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$. Then $\mathcal{P}$ is a subspace of $\mathbb{R}^3$ and these two vectors are a basis for $\mathcal{P}$.

Example: Find a basis for $S = \text{span}(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix})$.

Solution:
You can see by inspection that these vectors aren't linearly independent: the third is the sum of the first two. So $S = \mathrm{span}(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix})$. These two vectors are linearly independent, so they form a basis for S.

In more complicated situations, there are two ways to find a basis of the span of a set of vectors. The first way uses the following result:

Theorem 3.20: Let A and B be row equivalent matrices. Then $\mathrm{row}(A) = \mathrm{row}(B)$.

Proof: Suppose B is obtained from A by performing elementary row operations. Each of these operations expresses the new row as a linear combination of the previous rows. So every row of B is a linear combination of the rows of A. So $\mathrm{row}(B) \subseteq \mathrm{row}(A)$.

On the other hand, each row operation is reversible, so reversing the above argument gives that $\mathrm{row}(A) \subseteq \mathrm{row}(B)$. Therefore, $\mathrm{row}(A) = \mathrm{row}(B)$. $\quad \square$

This will be useful, because it is easy to understand the row space of a matrix in row echelon form.

Example: Let's redo the above example. Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ -2 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

whose rows are the given vectors. So $S = \mathrm{row}(A)$.

Row reduction produces the following matrix

$$B = \begin{bmatrix} 1 & 0 & 2/3 \\ 0 & 1 & 7/3 \\ 0 & 0 & 0 \end{bmatrix}$$

which is in reduced row echelon form. By Theorem 3.20, $S = \mathrm{row}(B)$. But the first two rows clearly give a basis for $\mathrm{row}(B)$, so another solution to the question is $\begin{bmatrix} 1 \\ 0 \\ 2/3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 7/3 \end{bmatrix}$.

Theorem: If R is a matrix in row echelon form, then the nonzero rows of R form a basis for $\mathrm{row}(R)$.

Example: Let

$$R = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_3 \\ \vec{a}_4 \end{bmatrix}$$

$\mathrm{row}(R)$ is the span of the nonzero rows, since zero rows don't contribute. So we just need to see that the nonzero rows are linearly independent. If we had $c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = \vec{0}$, then $c_1 = 0$, by looking at the first component. So $5c_2 = 0$, by looking at the second component. And so $8c_3 = 0$, by looking at the fourth component. So $c_1 = c_2 = c_3 = 0$.

The same argument works in general, by looking at the pivot (leading) columns, and this proves the Theorem.

This gives rise to the **row method** for finding a basis for a subspace S spanned by some vectors $\vec{v}_1, \ldots, \vec{v}_k$:

1. Form the matrix A whose rows are $\vec{v}_1, \ldots, \vec{v}_k$, so $S = \mathrm{row}(A)$.
2. Reduce A to row echelon form R.
3. The nonzero rows of R will be a basis of $S = \mathrm{row}(A) = \mathrm{row}(R)$.

Notice that the vectors you get are usually different from the vectors you started with. Given $S = \text{span}(\vec{v}_1, \ldots, \vec{v}_k)$, one can always find a basis for S which just omits some of the given vectors. We'll explain this next.

Suppose we form a matrix A whose columns are $\vec{v}_1, \ldots, \vec{v}_k$. A nonzero solution to the system $A\vec{x} = \vec{0}$ is exactly a dependency relationship between the given vectors. Also, recall that if R is row equivalent to A, then $R\vec{x} = \vec{0}$ has the same solutions as $A\vec{x} = \vec{0}$. This means that the columns of R have *the same* dependency relationships as the columns of A.

Example 3.47: Find a basis for the column space of

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

Solution: The reduced row echelon form is

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Write $\vec{r}_i$ for the columns of R and $\vec{a}_i$ for the columns of A. You can see immediately that $\vec{r}_3 = \vec{r}_1 + 2\vec{r}_2$ and $\vec{r}_5 = -\vec{r}_1 + 3\vec{r}_2 + 4\vec{r}_4$. So $\text{col}(R) = \text{span}(\vec{r}_1, \vec{r}_2, \vec{r}_4)$, and these three are linearly independent since they are standard unit vectors.

It follows that the columns of A have the same dependency relationships: $\vec{a}_3 = \vec{a}_1 + 2\vec{a}_2$ and $\vec{a}_5 = -\vec{a}_1 + 3\vec{a}_2 + 4\vec{a}_4$. Also, $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_4$ must be linearly independent. So a basis for $\text{col}(A)$ is given by $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_4$.

Note that these are the **columns corresponding to the leading 1's** in R!

Warning: Elementary row operations change the column space! So $\text{col}(A) \neq \text{col}(R)$. So while $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_4$ are a basis for $\text{col}(R)$, they are not a solution to the question asked.

The other kind of subspace that arises a lot is the **null space** of a matrix A, the subspace of solutions to the homogeneous system $A\vec{x} = \vec{0}$. We learned in Chapter 2 how to find a basis for this subspace, even though we didn't use this terminology.

Example 3.48: Find a basis for the null space of the 4×5 matrix A above.

Solution: The reduced row echelon form of $[A \mid \vec{0}\,]$ is

$$[R \mid \vec{0}\,] = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We see that x_3 and x_5 are free variables, so we let $x_3 = s$ and $x_5 = t$ and use back substitution to find that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix} \qquad \text{(See text.)}$$

Therefore, the two column vectors shown form a basis for the null space.

The vectors that arise in this way will always be linearly independent, since if all x_i's are 0, then the free variables must be zero, so the parameters must be zero.

Summary

Finding bases for $\operatorname{row}(A)$, $\operatorname{col}(A)$ and $\operatorname{null}(A)$:

1. Find the reduced row echelon form R of A.
2. The nonzero rows of R form a basis for $\operatorname{row}(A) = \operatorname{row}(R)$.
3. The columns of A that correspond to the columns of R with leading 1's

form a basis for $\mathrm{col}(A)$.
4. Use back substitution to solve $R\vec{x} = \vec{0}$; the vectors that arise are a basis for $\mathrm{null}(A) = \mathrm{null}(R)$.

You just need to do row reduction *once* to answer all three questions!

We have seen two ways to compute a basis of a span of a set of vectors. One is to make them the columns of a matrix, and the other is to make them the rows. The column method produces a basis using vectors from the original set. Both ways require about the same amount of work.

Similarly, if asked to find a basis for $\mathrm{row}(A)$, one could use the column method on A^T.

Dimension

We have seen that a subspace has many bases. Have you noticed anything about the number of vectors in each basis?

Theorem 3.23: Let S be a subspace of $\mathbb{R}^n$. Then any two bases for S have the same number of vectors.

Idea of proof:
Suppose that $\{\vec{u}_1, \vec{u}_2\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ were both bases for S. We'll show that this is impossible, by showing that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly dependent. Since $\{\vec{u}_1, \vec{u}_2\}$ is a basis, we can express each $\vec{v}_i$ in terms of the $\vec{u}_j$'s:

$$\begin{aligned}
\vec{v}_1 &= a_{11}\vec{u}_1 + a_{21}\vec{u}_2 \\
\vec{v}_2 &= a_{12}\vec{u}_1 + a_{22}\vec{u}_2 \\
\vec{v}_3 &= a_{13}\vec{u}_1 + a_{23}\vec{u}_2
\end{aligned}$$

Then

$$\begin{aligned}
& c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \\
= \; & c_1(a_{11}\vec{u}_1 + a_{21}\vec{u}_2) + c_2(a_{12}\vec{u}_1 + a_{22}\vec{u}_2) + c_3(a_{13}\vec{u}_1 + a_{23}\vec{u}_2) \\
= \; & (c_1 a_{11} + c_2 a_{12} + c_3 a_{13})\vec{u}_1 + (c_1 a_{21} + c_2 a_{22} + c_3 a_{23})\vec{u}_2
\end{aligned}$$

But the homogenous system

$$\begin{aligned} c_1 a_{11} + c_2 a_{12} + c_3 a_{13} &= 0 \\ c_1 a_{21} + c_2 a_{22} + c_3 a_{23} &= 0 \end{aligned}$$

has nontrivial solutions! (Why?) Therefore, we can find nontrivial c_1, c_2, c_3 such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \qquad \square$$

A very similar argument works for the general case.

Definition: The number of vectors in a basis for a subspace S is called the **dimension** of S, denoted $\dim S$.

Example: $\dim \mathbb{R}^n = n$

Example: If S is a line through the origin in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\dim S = 1$

Example: If S is a plane through the origin in $\mathbb{R}^3$, then $\dim S = 2$

Example: If $S = \text{span}(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix})$, then $\dim S = 2$