Math 1600A Lecture 2, Section 2, 8 Sept 2014

Announcements:

Read Section 1.2 for next class. Work through homework problems.

Lecture notes (this page) available from course web page. Also look for **announcements** there.

No tutorials this week. There is a quiz in tutorials next week.

Please **read over syllabus**, especially before e-mailing me with questions, as it covers all of the main points.

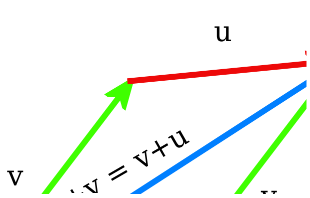
Let me know if the bookstore runs out of **texts or combo packs**.

Review of last lecture:

A vector can be represented by its list of components, e.g. $[1, 2, -1]$ is a vector in $\mathbb{R}^3$.
We write $\mathbb{R}^n$ for the set of all vectors with n real components, e.g. $[1, 2, 3, 4, 5, 6, 7]$ is in $\mathbb{R}^7$.

We also often write vectors as column vectors, e.g. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Vector addition: $[u_1, \ldots, u_n] + [v_1, \ldots, v_n] := [u_1 + v_1, \ldots, u_n + v_n]$.
E.g. $[3, 2, 1] + [1, 0, -1] = [4, 2, 0]$.



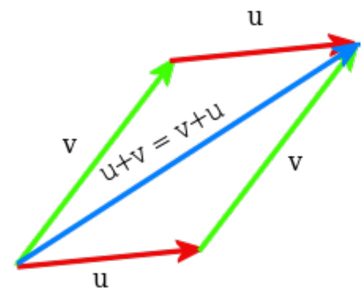
Scalar multiplication: $c[u_1, \ldots, u_n] := [cu_1, \ldots, cu_n]$.

E.g. $2[1, 2, 3, 4, 5] = [2, 4, 6, 8, 10]$.

Zero vector: $\vec{0} := [0, 0, \ldots, 0]$.

New material: Section 1.1, continued: Properties of vector operations

The picture to the right shows geometrically that vector addition is *commutative*: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.



In this true in $\mathbb{R}^n$? Let's check:

$$\begin{aligned} \vec{u} + \vec{v} &= [u_1 + v_1,\ \ldots,\ u_n + v_n] \\ &= [v_1 + u_1,\ \ldots,\ v_n + u_n] \\ &= \vec{v} + \vec{u}. \end{aligned}$$

Many other properties that hold for real numbers also hold for vectors: Theorem 1.1. But we'll see differences later.

Example: Simplification of an expression:

$$\begin{aligned} & 3\vec{b} + 2(\vec{a} - 4\vec{b}) \\ = \ & 3\vec{b} + 2\vec{a} - 8\vec{b} \\ = \ & 2\vec{a} - 5\vec{b} \end{aligned}$$

True/false: For every vector $\vec{u}$, we have $2\vec{u} = \vec{u} + \vec{u}$.

True, since both sides have components $[2u_1, \ldots, 2u_n]$. Or, from the distributive law (Theorem 1.1(f)) and Theorem 1.1(h), we have

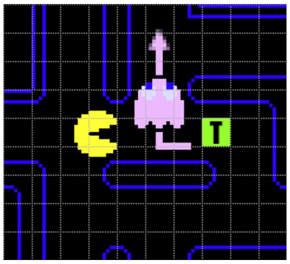
$$2\vec{u} = (1 + 1)\vec{u} = 1\vec{u} + 1\vec{u} = \vec{u} + \vec{u}.$$

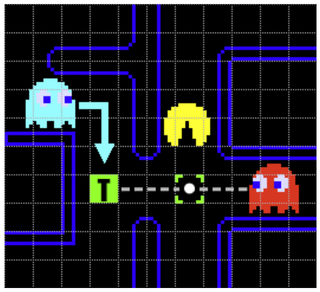
True/false: For every vector $\vec{u}$, we have $2\vec{u} \neq 3\vec{u}$.

False. If $\vec{u} = \vec{0}$, then $2\vec{u} = \vec{0} = 3\vec{u}$.

An important real-world application:

Pac-Man: Google's version, and How the ghosts move.





Derive an equation for Inky's target on board.

Linear combinations

Definition: A vector $\vec{v}$ is a **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there are scalars $c_1, c_2, \ldots, c_k$ so that

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k.$$

The numbers $c_1, \ldots, c_k$ are called the coefficients. They are not necessarily unique.

Example: Is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

Yes, since

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(Check!)}$$

Note: We also have

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(Check!)}$$

and many more possibilities.

We will learn later how to find all solutions.

Example: Is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

No, since any linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ has a zero as the second component.

Example: Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

Yes. The zero vector is a linear combination of *any* set of vectors, since you can just take $c_1 = c_2 = \cdots = c_k = 0$.

Coordinates

Example: Express $\vec{w}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ as a linear combination of $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We can solve this by using $\vec{u}$ and $\vec{v}$ to make a new coordinate system in the plane. Use the board to show that $\vec{w}_1 = 2\vec{u} + \vec{v}$.

Similarly, show that $\vec{w}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ can be expressed as $\vec{w}_2 = \vec{u} - 2\vec{v}$.

Note that in this case the coefficients are unique. In this situation, the coefficients are called the **coordinates** with respect to $\vec{u}$ and $\vec{v}$. So the coordinates of $\vec{w}_1$ with respect to $\vec{u}$ and $\vec{v}$ are 2 and 1, and the coordinates of $\vec{w}_2$ with respect to $\vec{u}$ and $\vec{v}$ are 1 and -2.

Working in a different coordinate system is a powerful tool.

Binary vectors

$\mathbb{Z}_2 := \{0, 1\}$, a set with two elements.

Multiplication is as usual.

Addition: $0 + 0 = 0,\ 0 + 1 = 1,\ 1 + 0 = 1,\ 1 + 1 = 0$.

$\mathbb{Z}_2^n :=$ vectors with n components in $\mathbb{Z}_2$.

E.g. $[0, 1, 1, 0, 1] \in \mathbb{Z}_2^5$.

$[0, 1, 1] + [1, 1, 0] = [1, 0, 1]$ in $\mathbb{Z}_2^3$.

There are 2^n vectors in $\mathbb{Z}_2^n$.