Math 1600 Lecture 21, Section 2, 24 Oct 2014

Announcements:

Read Markov chains part of Section 3.7 for next class. Work through recommended homework questions.

Extra Midterm Review: Today, 4:30-6:00pm, MC105B. Bring questions.

Midterm: Saturday, October 25, 7-10pm. Rooms, based on first letter of last name: A-E: UCC37. F-Ma: UCC56 (this room). Mc-Z: UCC146. Be sure to write in the correct room! It will cover the material up to and including Monday's lecture. **Review the policies about illness on course website.**

Help Centers: Monday-Friday 2:30-6:30 in MC106.

Last class, we finished Section 3.5. That was a key section, so please study it carefully. We won't use that material today, so I will jump right into Section 3.6.

Section 3.6: Linear Transformations

Given an $m \times n$ matrix A, we can use A to transform a column vector in $\mathbb{R}^n$ into a column vector in $\mathbb{R}^m$. We write:

$$T_A(\vec{x}) = A\vec{x} \quad \text{for } \vec{x} \text{ in } \mathbb{R}^n$$

Example: If $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$ then

$$T_A\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = A\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

In general,

$$T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} y \\ 2x+3y \\ 4x+5y \end{bmatrix}$$

Note that the matrix A is visible in the last expression.

Here is an applet giving many examples.

Any rule T that assigns to each $\vec{x}$ in $\mathbb{R}^n$ a unique vector $T(\vec{x})$ in $\mathbb{R}^m$ is called a **transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$ and is written $T : \mathbb{R}^n \to \mathbb{R}^m$.

For our A above, we have $T_A : \mathbb{R}^2 \to \mathbb{R}^3$. T_A is in fact a *linear* transformation.

Definition: A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if:
1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, and
2. $T(c\vec{u}) = c\,T(\vec{u})$ for all $\vec{u}$ in $\mathbb{R}^n$ and all scalars c.

You can check directly that our T_A is linear. For example,

$$T_A\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T_A\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = \begin{bmatrix} cy \\ 2cx+3cy \\ 4cx+5cy \end{bmatrix} = c\begin{bmatrix} y \\ 2x+3y \\ 4x+5y \end{bmatrix} = c\,T_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$

Check condition (1) yourself, or see Example 3.55.

In fact, *every* T_A is linear:

Theorem 3.30: Let A be an $m \times n$ matrix. Then $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Proof: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$ and let $c \in \mathbb{R}$. Then

$$T_A(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = T_A(\vec{u}) + T_A(\vec{v})$$

and

$$T_A(c\vec{u}) = A(c\vec{u}) = c\,A\vec{u} = c\,T_A(\vec{u}) \qquad \square$$

Example 3.56: Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that sends each point to its reflection in the x-axis. Show that F is linear.

Solution: We need to show that

$$F(\vec{u} + \vec{v}) = F(\vec{u}) + F(\vec{v}) \quad \text{and} \quad F(c\vec{u}) = c\,F(\vec{u})$$

Give a geometrical explanation on the board.

Algebraically, note that $F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ -y \end{bmatrix}$, from which you can check directly that F is linear. (Exercise.)

Or, observe that $F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, so $F = T_A$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Example: Let $N : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation

$$N\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} xy \\ x + y \end{bmatrix}$$

Is N linear?

Solution: No. For example, $N\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ but $N\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \neq 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

It turns out that *every* linear transformation is a matrix transformation.

Theorem 3.31: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T = T_A$, where

$$A = [\, T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n) \,]$$

Proof: We just check:

$$\begin{aligned} T(\vec{x}) &= T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) \\ &= x_1 T(\vec{e}_1) + \cdots + x_n T(\vec{e}_n) \quad \text{since } T \text{ is linear} \\ &= [\, T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n) \,] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= A\vec{x} = T_A(\vec{x}) \qquad \square \end{aligned}$$

The matrix A is called the **standard matrix** of T and is written $[T]$.

Example: Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + 3y - z \\ y + z \end{bmatrix}.$$

Is T linear? If so, find $[T]$. On board.

Example 3.58: Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by an angle θ counterclockwise about the origin. Show that R_θ is linear and find its standard matrix.

Solution: We need to show that

$$R_\theta(\vec{u} + \vec{v}) = R_\theta(\vec{u}) + R_\theta(\vec{v}) \quad \text{and} \quad R_\theta(c\vec{u}) = c\,R_\theta(\vec{u})$$

A geometric argument shows that R_θ is linear. On board.

To find the standard matrix, we note that

$$R_\theta\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \text{and} \qquad R_\theta\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Therefore, the standard matrix of R_θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Now that we know the matrix, we can compute rotations of arbitrary vectors. For example, to rotate the point $(2, -1)$ by 60°:

$$R_{60}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} (2+\sqrt{3})/2 \\ (2\sqrt{3}-1)/2 \end{bmatrix}$$

Rotations will be one of our main examples.

The applet gives examples involving rotations.

New linear transformations from old

If $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^p$, then $S(T(\vec{x}))$ makes sense for $\vec{x}$ in $\mathbb{R}^m$. The **composition** of S and T is the transformation $S \circ T : \mathbb{R}^m \to \mathbb{R}^p$ defined by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})).$$

If S and T are linear, it is easy to check that this new transformation $S \circ T$ is automatically linear. For example,

$$(S \circ T)(\vec{u} + \vec{v}) = S(T(\vec{u} + \vec{v})) = S(T(\vec{u}) + T(\vec{v}))$$
$$= S(T(\vec{u})) + S(T(\vec{v})) = (S \circ T)(\vec{u}) + (S \circ T)(\vec{v}).$$

Any guesses for how the the matrix for $S \circ T$ is related to the matrices for S and T?

Theorem 3.32: $[S \circ T] = [S][T]$, where $[\ \ ]$ is used to denote the matrix of a linear transformation.

Proof: Let $A = [S]$ and $B = [T]$. Then

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) = S(B\vec{x}) = A(B\vec{x}) = (AB)\vec{x}$$

so $[S \circ T] = AB$. $\square$

It's because of this that matrix multiplication is defined how it is! Notice also that the condition on the sizes of matrices in a product matches the requirement that S and T be composable.

Example 3.61: Find the standard matrix of the transformation that rotates 90° counterclockwise and then reflects in the x-axis. How do $F \circ R_{90}$ and $R_{90} \circ F$ compare? On board.