Math 1600 Lecture 22, Section 2, 27 Oct 2014

Announcements:

Read Sections 4.0 and 4.1 for next class. Work through recommended homework questions.

Midterm results: Grades will be posted on OWL over the next few days. Midterms will be returned at the tutorials **next** week.

Tutorials: No tutorials this week! Fall break Thurs/Fri.

Office hour: today, 3:00-3:30, MC103B.

Help Centers: Monday-Friday 2:30-6:30 in MC 106 except Thurs/Fri.

Review of last lecture: Section 3.6: Linear Transformations

Given an $m \times n$ matrix A, we can use A to transform a column vector in $\mathbb{R}^n$ into a column vector in $\mathbb{R}^m$. We write:

$$T_A(\vec{x}) = A\vec{x} \quad \text{for } \vec{x} \text{ in } \mathbb{R}^n$$

Any rule T that assigns to each $\vec{x}$ in $\mathbb{R}^n$ a unique vector $T(\vec{x})$ in $\mathbb{R}^m$ is called a **transformation** from $\mathbb{R}^n$ to $\mathbb{R}^m$ and is written $T : \mathbb{R}^n \to \mathbb{R}^m$.

Definition: A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if:
1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, and
2. $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ in $\mathbb{R}^n$ and all scalars c.

Theorem 3.30: Let A be an $m \times n$ matrix. Then $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

Theorem 3.31: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $T = T_A$, where

$$A = [\,T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_n)\,]$$

The matrix A is called the **standard matrix** of T and is written $[T]$.

Example 3.58: Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by an angle θ counterclockwise about the origin. Show that R_θ is linear and find its standard matrix.

Solution: A geometric argument shows that R_θ is linear.

To find the standard matrix, we note that

$$R_\theta\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \qquad \text{and} \qquad R_\theta\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

Therefore, the standard matrix of R_θ is $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$.

New linear transformations from old

If $T : \mathbb{R}^m \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^p$, then $S(T(\vec{x}))$ makes sense for $\vec{x}$ in $\mathbb{R}^m$. The **composition** of S and T is the transformation $S \circ T : \mathbb{R}^m \to \mathbb{R}^p$ defined by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})).$$

If S and T are linear, it is easy to check that this new transformation $S \circ T$ is automatically linear.

Theorem 3.32: $[S \circ T] = [S][T]$.

We saw an applet illustrating linear transformations.

New material

Example: It is geometrically clear that $R_\theta \circ R_\phi = R_{\theta+\phi}$. This tells us that

$$\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

This implies some trigonometric identities. For example, looking at the top-left entry, we find that

$$\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

Other trig identities also follow.

Note that R_0 is rotation by zero degrees, so $R_0(\vec{x}) = \vec{x}$. We say that R_0 is the **identity transformation**, which we write $I : \mathbb{R}^2 \to \mathbb{R}^2$. Similarly, $R_{360} = I$.

Since $R_{120} \circ R_{120} \circ R_{120} = R_{360} = I$, we must have $[R_{120}]^3 = [I] = I$. This is how I came up with the answer $[R_{120}] = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$ to the challenge problem in Lecture 15.

Our new point of view about matrix multiplication gives us a **geometrical** way to understand it!

Inverses of Linear Transformations

Since $R_{60} \circ R_{-60} = R_0 = I$, it follows that $[R_{60}][R_{-60}] = I$. So $[R_{-60}] = [R_{60}]^{-1}$. See Example 3.62 for details.

Definition: Let S and T be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then S and T are **inverse transformations** if $S \circ T = I$ and $T \circ S = I$. When this is the case, we say that S and T are **invertible** and are **inverses**.

The same argument as for matrices shows that an inverse is unique when it exists, so we write $S = T^{-1}$ and $T = S^{-1}$.

Theorem 3.33: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then T is invertible if and only if $[T]$ is an invertible matrix. In this case, $[T^{-1}] = [T]^{-1}$.

The argument is easy and is essentially what we did for R_{60}.

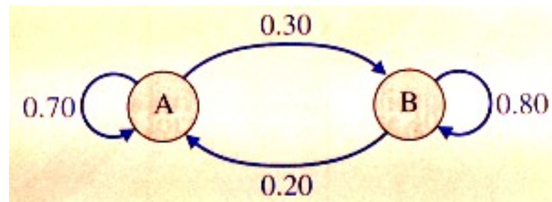
Question: Is projection onto the x-axis invertible?

Question: Is reflection in the x-axis invertible?

Question: Is translation a linear transformation?

Section 3.7: Markov Chains

Example 3.64: 200 people are testing two brands of toothpaste, Brand A and Brand B. Each month they are allowed to switch brands. The research firm observes the following:



- Of those using Brand A in a given month, 70% continue in the following month and 30% switch to B.
- Of those using Brand B in a given month, 80% continue in the following month and 20% switch to A.

This is called a **Markov chain**. There are definite states, and from each state there is a **transition probability** for moving to another state at each time step. These probabilities are constant and depend only on the current state.

Suppose at the start that 120 people use Brand A and 80 people use Brand B. Then, in the next month,

$$0.70(120) + 0.20(80) = 100 \quad \text{will use Brand A}$$

and

$$0.30(120) + 0.80(80) = 100 \quad \text{will use Brand B}$$

This is a matrix equation:

$$\begin{bmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{bmatrix} \begin{bmatrix} 120 \\ 80 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

Write P for the **transition matrix** and $\vec{x}_k$ for the **state vector** after k months have gone by. Then $\vec{x}_{k+1} = P\vec{x}_k$. So

$$\vec{x}_2 = P\vec{x}_1 = \begin{bmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 90 \\ 110 \end{bmatrix}$$

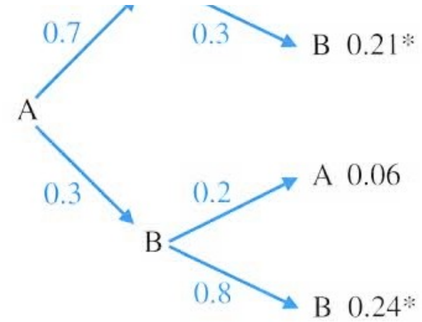
and we see that there are 90 people using Brand A and 110 using Brand B after 2 months.

We can also work with the percentage of people using each brand. Then $\vec{x}_0 = \begin{bmatrix} 120/200 \\ 80/200 \end{bmatrix} = \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}$ and $P\vec{x}_0 = \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix}$. Vectors with non-negative components that sum to 1 are called **probability vectors**

Note that P is a **stochastic matrix**: this means that it is square and that each column is a probability vector.

The column indices of P correspond to the current state and the row indices correspond to the next state. The entry P_{ij} is the probability that you transition from state j to state i in one time step, where we now label the states with numbers.

Multiple steps: Can we compute the probability that we go from state j to state i in **two** steps?



Well, $x_{k+2} = Px_{k+1} = P^2 x_k$, so the matrix P^2 computes this transition:

$$P^2 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.55 & 0.30 \\ 0.45 & 0.70 \end{bmatrix}$$

So the probability of going from Brand A to Brand B after two steps is $(P^2)_{21} = 0.45 = 0.21 + 0.24$.

More generally, $(P^k)_{ij}$ is the probability of going from state j to state i in k steps.

Long-term behaviour: By multiplying by P, you can show that the state evolves as follows:

$$\begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix}, \begin{bmatrix} 0.50 \\ 0.50 \end{bmatrix}, \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix}, \begin{bmatrix} 0.425 \\ 0.575 \end{bmatrix}, \begin{bmatrix} 0.412 \\ 0.588 \end{bmatrix}, \begin{bmatrix} 0.406 \\ 0.594 \end{bmatrix},$$
$$\begin{bmatrix} 0.403 \\ 0.597 \end{bmatrix}, \begin{bmatrix} 0.402 \\ 0.598 \end{bmatrix}, \begin{bmatrix} 0.401 \\ 0.599 \end{bmatrix}, \begin{bmatrix} 0.400 \\ 0.600 \end{bmatrix}, \begin{bmatrix} 0.400 \\ 0.600 \end{bmatrix}, \ldots$$

with 40% of the people using Brand A in the long run. Since

$$\begin{bmatrix} 0.70 & 0.20 \\ 0.30 & 0.80 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix},$$

once we reach this state, we don't leave. A state $\vec{x}$ with $P\vec{x} = \vec{x}$ is called a **steady state vector**. We'll prove below that every Markov chain has a steady state vector!

Here's how to find it. We want to find $\vec{x}$ such that $(I - P)\vec{x} = \vec{0}$. The augmented system is

$$[I - P \mid \vec{0}\,] = \left[\begin{array}{cc|c} 0.30 & -0.20 & 0 \\ -0.30 & 0.20 & 0 \end{array}\right]$$

which row reduces to

$$\left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is

$$x_1 = \frac{2}{3}\,t, \quad x_2 = t$$

We'd like a probability vector, so $\frac{2}{3}\,t + t = 1$ which means that $t = 3/5$.
This gives $\vec{x} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$ as we found above.

Theorem: Every Markov chain has a non-trivial steady state vector.

This appears in the book as Theorem 4.30 in Section 4.6.

Proof: Let P be the transition matrix. We want to find a non-trivial solution to $(I - P)\vec{x} = \vec{0}$. By the fundamental theorem of invertible matrices and the fact that $\mathrm{rank}(I - P) = \mathrm{rank}((I - P)^T)$, this is equivalent to $(I - P)^T\vec{x} = \vec{0}$ having a non-trivial solution. That is, finding a non-trivial $\vec{x}$ such that

$$P^T\vec{x} = \vec{x} \quad (\text{since } I^T = I).$$

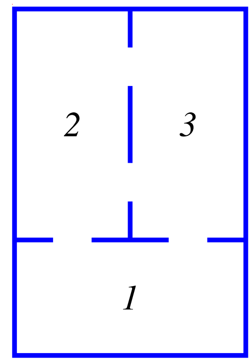
But since P is a stochastic matrix, we always have

$$P^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

So therefore $P\vec{x} = \vec{x}$ also has a (different) non-trivial solution. □

Example 3.65: A Markov chain can have more than two states. A rat is in a maze with three rooms, and always chooses to go through one of the doors with equal probability. Draw the state diagram, determine the transition matrix P and describe how to find a steady-state vector.

Solution: Draw state diagram on board.



From this, we find the transition matrix

$$P = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & 0 & 2/3 \\ 1/2 & 2/3 & 0 \end{bmatrix}$$

The P_{ij} entry is the probability of going from room j to room i.

A **steady state vector** is a vector $\vec{x}$ such that $P\vec{x} = \vec{x}$. That is, $\vec{x} - P\vec{x} = \vec{0}$, or $(I - P)\vec{x} = \vec{0}$. To find a non-trivial steady state vector for this Markov chain, we solve the homogeneous system with coefficient matrix $I - P$:

$$\left[\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 0 \\ -1/2 & 1 & -2/3 & 0 \\ -1/2 & -2/3 & 1 & 0 \end{array}\right]$$

In RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $x_3 = t$, $x_2 = t$ and $x_1 = \frac{2}{3}\,t$. If we want a probability vector, then we want $t + t + \frac{2}{3}\,t = 1$, so $t = 3/8$, so we get $\begin{bmatrix} 2/8 \\ 3/8 \\ 3/8 \end{bmatrix}$.

We'll probably study Markov chains again in Section 4.6.