

## Math 1600 Lecture 25, Section 2, 5 Nov 2014

### Announcements:

Today we mostly finish 4.2. **Read** Section 4.3 and Appendix D (polynomials) for next class. Work through recommended [homework questions](#).

**Drop date:** Wednesday, November 5 (today!)

**Tutorials:** Quiz this week covers 3.6, 3.7 (Markov chains) and 4.1. Midterms returned in tutorials. [Solutions](#) available. Average:  $47.5/70 = 68\%$ .

**Office hour:** Wednesday, 11:30-noon, MC103B.

**Help Centers:** Monday-Friday 2:30-6:30 in MC106.

### Summary of Section 4.2: Determinants

For an  $n \times n$  matrix  $A$ , write  $A_{ij}$  for the matrix obtained from  $A$  by deleting the  $i$ th row and the  $j$ th column. Then  $\det A_{ij}$  is called the  $(i, j)$ -**minor** of  $A$ , and

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

is called the  $(i, j)$ -**cofactor** of  $A$ .

**Definition:** Let  $A = [a_{ij}]$  be an  $n \times n$  matrix. Then the **determinant** of  $A$  is the scalar

$$\begin{aligned} \det A = |A| &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= \sum_{j=1}^n a_{1j}C_{1j} = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det A_{1j}. \end{aligned}$$

We define the determinant of a  $1 \times 1$  matrix  $[a]$  to be  $a$ .

This is a recursive definition!

For  $n = 2$ :

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ = a_{11}|a_{22}| - a_{12}|a_{21}| = a_{11}a_{22} - a_{12}a_{21},$$

as we defined earlier.

For a  $3 \times 3$  matrix  $A$ , we have

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The computation can be very long if there aren't many zeros! We'll learn some better methods.

The above is called the **cofactor expansion along the first row**. It turns out that *any* row or column works!

**Theorem 4.1 (The Laplace Expansion Theorem):** Let  $A$  be any  $n \times n$  matrix. Then for each  $i$  we have

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^n a_{ij}C_{ij}$$

(**cofactor expansion along the  $i$ th row**). And for each  $j$  we have

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^n a_{ij}C_{ij}$$

(**cofactor expansion along the  $j$ th column**).

The book proves this result at the end of this section, but we won't cover the proof.

The signs in the cofactor expansion form a checkerboard pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A **triangular** matrix is a square matrix that is all zero below the diagonal or above the diagonal.

**Theorem 4.2:** If  $A$  is triangular, then  $\det A$  is the product of the diagonal entries.

## Better methods

Laplace Expansion is convenient when there are appropriately placed zeros in the matrix, but it is not good in general. It produces  $n!$  different terms, which is waaaaay too slow for large matrices.

So how do we do better? Like always, we turn to row reduction! These properties will be what we need:

**Theorem 4.3:** Let  $A$  be a square matrix.

- a. If  $A$  has a zero row, then  $\det A = 0$ .
- b.** If  $B$  is obtained from  $A$  by interchanging two rows, then  $\det B = -\det A$ .
- c. If  $A$  has two identical rows, then  $\det A = 0$ .
- d.** If  $B$  is obtained from  $A$  by multiplying a row of  $A$  by  $k$ , then  $\det B = k \det A$ .
- e. If  $A$ ,  $B$  and  $C$  are identical in all rows except the  $i$ th row, and the  $i$ th row of  $C$  is the sum of the  $i$ th rows of  $A$  and  $B$ , then  $\det C = \det A + \det B$ .
- f.** If  $B$  is obtained from  $A$  by adding a multiple of one row to another, then  $\det B = \det A$ .

All of the above statements are true with rows replaced by columns.

The bold statements are the ones that are useful for understanding how row operations change the determinant.

## New material: Section 4.2: Determinants (cont)

**Example:** Use row operations to compute  $\det A$  by reducing to triangular form,

$$\text{where } A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 2 & 1 & 2 \\ 2 & 2 & 12 & 8 \\ 1 & 2 & 3 & 9 \end{bmatrix}.$$

**Solution:**

$$\begin{array}{l} \begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 2 & 1 & 2 \\ 2 & 2 & 12 & 8 \\ 1 & 2 & 3 & 9 \end{bmatrix} \xrightarrow{(1/2)R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \\ 2 & 2 & 12 & 8 \\ 1 & 2 & 3 & 9 \end{bmatrix} \\ \begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - R_1 \end{array} \xrightarrow{\quad} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \end{array}$$

The determinant of the last matrix is  $(1)(-2)(-2)(5) = 20$ , so the determinant of the original matrix is  $-40$

**Example:** Same for  $A = \begin{bmatrix} 2 & 3 & -1 \\ -4 & -6 & 2 \\ 2 & 5 & 3 \end{bmatrix}$ . On board.

Note that you can even mix and match row and column operations, if it simplifies the work.

**Question:** What happens if you do something like "replace  $R_3$  with  $R_1 + 3R_3$ "?

Row reduction of an  $n \times n$  matrix requires roughly  $n^3$  operations in general, which is **much** less than  $n$  factorial. E.g. 3 hours vs. 1 year to do a  $10 \times 10$  matrix by hand?

Use **Laplace expansion** for matrices with lots of zeros, or matrices with variables like  $\lambda$ . Use **row reduction** for  $4 \times 4$  or larger matrices with not many zeros. For a  $3 \times 3$  matrix, which is better depends on the matrix.

## Determinants and Invertibility

**Theorem 4.6:** A square matrix  $A$  is invertible if and only if  $\det A \neq 0$ .

The book proves this using elementary matrices, which we aren't covering, but here is a simpler proof.

**Proof:** If  $A$  is invertible, then by the Fundamental Theorem, the reduced row echelon form of  $A$  is  $I$ . Each elementary row operation either leaves the determinant the same or multiplies by a non-zero number. Since  $\det I = 1 \neq 0$ , we must also have  $\det A \neq 0$ .

On the other hand, if  $A$  is not invertible, then the reduced row echelon form  $R$  has a zero row, so  $\det R = 0$ . Again,  $\det A = k \det R$  for some  $k$ , so  $\det A = 0$  too.

□

**Example:** The  $4 \times 4$  matrix above is invertible, but the  $3 \times 3$  is not. The computations illustrate the proof of the theorem.

**Question:** Is the matrix  $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 4 & 7 & 8 & 9 \end{bmatrix}$  invertible?

## Determinants and Matrix Operations

**Theorems 4.7 to 4.10:** Let  $A$  be an  $n \times n$  matrix. Then:

$$4.7: \det(kA) = k^n \det A$$

$$4.8: \det(AB) = (\det A)(\det B)$$

$$4.9: \det(A^{-1}) = \frac{1}{\det A}, \text{ if } A \text{ is invertible.}$$

$$4.10: \det(A^T) = \det A$$

**Note:** There is **no formula** for  $\det(A + B)$ .

### Proofs:

**4.7:** Follows from Theorem 4.3(d), since each row is multiplied by  $k$ .

**4.8:** The book uses elementary matrices to prove this, which we haven't covered, so there is no easy way for us to prove this. I'll do an example below.

**4.9:** This follows from 4.8. Since  $\det(AA^{-1}) = \det(I) = 1$  we have  $(\det A)(\det A^{-1}) = 1$ , so  $\det A^{-1} = 1/\det A$ .

**4.10:** Computing  $\det A$  using expansion along the first row produces the same thing as computing  $\det(A^T)$  by expanding along the first column. (Proof by

induction on the size of the matrix.)  $\square$

**Example:** Illustrate all four statements with  $2 \times 2$  matrices, on board.

**True/false:** If  $A$  and  $B$  are invertible, then so is  $A + B$ .

**True/false:** If  $A$  is not invertible, then  $AB$  is not invertible.

## Cramer's Rule

Cramer's Rule is a formula for solving a system of  $n$  equations in  $n$  unknowns. It is not efficient computationally, but is useful theoretically.

**Notation:** If  $A$  is an  $n \times n$  matrix and  $\vec{b} \in \mathbb{R}^n$ , we write  $A_i(\vec{b})$  for the matrix obtained from  $A$  by replacing the  $i$ th column with the vector  $\vec{b}$ :

$$A_i(\vec{b}) = [\vec{a}_1 \cdots \vec{a}_{i-1} \vec{b} \vec{a}_{i+1} \cdots \vec{a}_n]$$

**Theorem:** Let  $A$  be an **invertible**  $n \times n$  matrix and let  $\vec{b}$  be in  $\mathbb{R}^n$ . Then the unique solution  $\vec{x}$  of the system  $A\vec{x} = \vec{b}$  has components

$$x_i = \frac{\det(A_i(\vec{b}))}{\det A}, \quad \text{for } i = 1, \dots, n$$

**Example 4.16:** On board: Use Cramer's rule to solve

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ -x_1 + 4x_2 &= 1 \end{aligned}$$

**Proof:**

Suppose  $A\vec{x} = \vec{b}$ . Consider  $I_i(\vec{x}) = [\vec{e}_1 \cdots \vec{x} \cdots \vec{e}_n]$ . Then

$$\begin{aligned} A I_i(\vec{x}) &= A [\vec{e}_1 \cdots \vec{x} \cdots \vec{e}_n] = [A\vec{e}_1 \cdots A\vec{x} \cdots A\vec{e}_n] \\ &= [\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n] = A_i(\vec{b}). \end{aligned}$$

So

$$(\det A)(\det I_i(\vec{x})) = \det(A I_i(\vec{x})) = \det(A_i(\vec{b})).$$

But

$$\det I_i(\vec{x}) = \begin{vmatrix} 1 & \cdots & x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & x_i & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & x_n & \cdots & 1 \end{vmatrix} = x_i$$

by expanding along  $i$ th row. So the claim follows.  $\square$

**Note:** This is **not** an efficient method. For an  $n \times n$  system, you have to compute  $n + 1$  determinants. But the work in computing 1 determinant is enough to solve the system by our usual method.

## Matrix Inverse using the Adjoint

The matrix

$$\text{adj}A := [C_{ji}] = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

is called the **adjoint** of  $A$ .

**Theorem:** If  $A$  is an invertible matrix, then

$$A^{-1} = \frac{1}{\det A} \text{adj}A$$

On board: explain the  $2 \times 2$  special case.

I will explain the general case next lecture.