Math 1600 Lecture 26, Section 2, 7 Nov 2014

Announcements:

Today we finish Section 4.2, discuss Appendix D and start Section 4.3. Continue **reading** Section 4.3 and Appendix D for next class. We'll also learn how Google ranks page next class. Work through recommended homework questions.

Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Review Questions

True/false: $\det(AB) = (\det A)(\det B)$.

True/false: $\det(A+B) = \det A + \det B$.

Question: $\det(3I_2) = 3^2 \det I_2 = 3^2 = 9$

Question: $\begin{vmatrix} 0 & 0 & a \\ 0 & b & c \\ d & e & f \end{vmatrix} = - \begin{vmatrix} d & e & f \\ 0 & b & c \\ 0 & 0 & a \end{vmatrix} = -abd \quad \text{(not triangular!)}$

Partial review of last class: Cofactors and Cramer's Rule

For an $n \times n$ matrix A, write A_{ij} for the matrix obtained from A by deleting the ith row and the jth column. Then $\det A_{ij}$ is called the (i,j)**-minor** of A, and

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

is called the (i,j)**-cofactor** of A.

Notation: If A is an $n \times n$ matrix and $\vec{b} \in \mathbb{R}^n$, we write $A_i(\vec{b})$ for the matrix obtained from A by replacing the ith column with the vector $\vec{b}$:

$$A_i(\vec{b}) = [\vec{a}_1 \cdots \vec{a}_{i-1}\, \vec{b}\, \vec{a}_{i+1} \cdots \vec{a}_n\,]$$

Theorem 4.11: Let A be an invertible $n \times n$ matrix and let $\vec{b}$ be in $\mathbb{R}^n$. Then the unique solution $\vec{x}$ of the system $A\vec{x} = \vec{b}$ has components

$$x_i = \frac{\det(A_i(\vec{b}))}{\det A}\,, \quad \text{for } i = 1, \ldots, n$$

New material: Matrix Inverse using the Adjoint

Suppose A is invertible. We'll use Cramer's rule to find a formula for $X = A^{-1}$. We know that $AX = I$, so the jth column of X satisfies $A\vec{x}_j = \vec{e}_j$. By Cramer's Rule,

$$x_{ij} = \frac{\det(A_i(\vec{e}_j))}{\det A}$$

By expanding along the ith column, we see that

$$\det(A_i(\vec{e}_j)) = C_{ji}$$

So

$$x_{ij} = \frac{1}{\det A}\,C_{ji}, \quad \text{i.e.,} \quad X = \frac{1}{\det A}\,[C_{ij}]^T$$

The matrix

$$\text{adj}A := [C_{ji}] = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

is called the **adjoint** of A.

Theorem 4.12: If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det A}\,\text{adj}A$$

Example: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the cofactors are

$$C_{11} = +\det[d] = +d \qquad C_{12} = -\det[c] = -c$$
$$C_{21} = -\det[b] = -b \qquad C_{22} = +\det[a] = +a$$

so the adjoint matrix is

$$\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and

$$A^{-1} = \frac{1}{\det A}\,\text{adj}A = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

as we saw before.

See Example 4.17 in the text for a 3×3 example. This is not generally a good computational approach. It's importance is theoretical.

Appendix D: Polynomials

You should read this Appendix yourself. I will cover it briefly.

A **polynomial** is a function p of a single variable x that can be written in the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where the **coefficients** a_i are constants. The highest power of x appearing with a non-zero coefficient is called the **degree** of p.

Examples: $2 - 0.5x + \sqrt{2}x^3$, $\ln\left(\dfrac{e^{5x^3}}{e^{3x}}\right) = \ln(e^{5x^3 - 3x}) = 5x^3 - 3x$

Non-examples: $\sqrt{x}$, $1/x$, $\cos(x)$, $\ln(x)$.

(The text gives more examples, non-examples and explanations.)

Addition of polynomials is easy:

$$(1 + 2x - 4x^3) + (3 - 3x^2 + 6x^3) = 4 + 2x - 3x^2 + 2x^3$$

To **multiply** polynomials, you use the distributive law and collect terms:

$$\begin{aligned}(x+3)(1+2x+4x^2) &= x(1+2x+4x^2) + 3(1+2x+4x^2) \\ &= x + 2x^2 + 4x^3 + 3 + 6x + 12x^2 \\ &= 3 + 7x + 14x^2 + 4x^3\end{aligned}$$

Note that $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$.

If f and g are polynomials, sometimes you can find a polynomial q such that $f(x) = g(x)q(x)$, and sometimes you can't. If you can, then we say that g is a **factor** of f.

Example: Is $(x - 2)$ a factor of $x^2 - x - 2$?

Solution: If it is, then the quotient has degree 1. So suppose $x^2 - x - 2 = (x - 2)(ax + b)$. Then $ax^2 = x^2$, so $a = 1$. And $-x = -2ax + bx = -2x + bx$, so $b = 1$. Check the constant term: $-2 = -2b$. It works, so $x^2 - x - 2 = (x - 2)(x + 1)$, and the answer is "yes".

Example: Is $(x - 2)$ a factor of $x^2 + x - 2$?

Solution: If it is, then the quotient has degree 1. So suppose $x^2 + x - 2 = (x - 2)(ax + b)$. Then $ax^2 = x^2$, so $a = 1$. And $x = -2ax + bx = -2x + bx$, so $b = 3$. Check the constant term: $-2 = -2b$. Nope, so the answer is "no".

The above ad hoc method works for a degree 1 polynomial. For higher degrees, one can use long division (see Example D.4). But the degree 1 case will be most important to us, and is made even simpler by the following result:

Theorem D.2 (The Factor Theorem): Let f be a polynomial and let a be a constant. Then $f(a) = 0$ if and only if $x - a$ is a factor of $f(x)$.

When $f(a) = 0$, we say that a is a **zero** of f or a **root** of f.

It is clear that if $f(x) = (x - a)q(x)$, then $f(a) = 0$. The book explains the other direction.

Once you find a zero, you can use the ad hoc method shown above to find the other factor q. We'll see more examples soon.

Our interest will be in finding all zeros of a polynomial f of degree n. By the above, if you find a zero a, then $f(x) = (x-a)q(x)$, where q has degree $n-1$. If there is another root b of f, it must be a root of q, and so q will factor as $q(x) = (x-b)r(x)$, where r has degree $n-2$. Since the degrees are going down by one, there can be at most n distinct roots in total:

Theorem D.4 (The Fundamental Theorem of Algebra): A polynomial of degree n has at most n distinct roots.

Section 4.3: Eigenvalues and Eigenvectors

Recall from Section 4.1:

Definition: Let A be an $n \times n$ matrix. A scalar λ (lambda) is called an **eigenvalue** of A if there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Such a vector $\vec{x}$ is called an **eigenvector** of A corresponding to λ.

The eigenvectors for a given eigenvalue λ are the **nonzero** solutions to $(A - \lambda I)\vec{x} = \vec{0}$.

Definition: The collection of **all** solutions to $(A - \lambda I)\vec{x} = \vec{0}$ is a subspace called the **eigenspace** of λ and is denoted E_λ. In other words,

$$E_\lambda = \mathrm{null}(A - \lambda I).$$

It consists of the eigenvectors plus the zero vector.

By the fundamental theorem of invertible matrices, $A - \lambda I$ has a nontrivial null space if and only if it is not invertible, and we now know that this is the case if and only if $\det(A - \lambda I) = 0$.

The expression $\det(A - \lambda I)$ is always a polynomial in λ. For example, when $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \end{aligned}$$

If A is 3×3, then $\det(A - \lambda I)$ is equal to

$$(a_{11} - \lambda) \begin{vmatrix} a_{22} - \lambda & a_{23} \\ a_{32} & a_{33} - \lambda \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} - \lambda \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} - \lambda \\ a_{31} & a_{32} \end{vmatrix}$$

which is a degree 3 polynomial in λ.

Similarly, if A is $n \times n$, $\det(A - \lambda I)$ will be a degree n polynomial in λ. It is called the **characteristic polynomial** of A, and $\det(A - \lambda I) = 0$ is called the **characteristic equation**.

Finding eigenvalues and eigenspaces: Let A be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$.
2. Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$.
3. For each eigenvalue λ, find a basis for $E_\lambda = \text{null}(A - \lambda I)$ by solving the system $(A - \lambda I)\vec{x} = \vec{0}$.

So we need to get good at solving polynomial equations. Solutions are called **zeros** or **roots**. We saw above that a degree n polynomial has at most n distinct roots. Therefore:

Theorem: An $n \times n$ matrix A has at most n distinct eigenvalues.

Example 4.18: Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$.

Solution: 1. On board, compute the characteristic polynomial:

$$\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$$

2. To find the roots, it is often worth trying a few small integers to start. We see that $\lambda = 1$ works. So by the factor theorem, we know $\lambda - 1$ is a factor:

$$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = (\lambda - 1)(-\lambda^2 + 3\lambda - 2)$$

Now we need to find roots of $-\lambda^2 + 3\lambda - 2$. Again, $\lambda = 1$ works, and this factors as $-(\lambda - 1)(\lambda - 2)$. So

$$\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2)$$

and the roots are $\lambda = 1$ and $\lambda = 2$.

3. To find the $\lambda = 1$ eigenspace, we do row reduction:

$$[A - I \mid 0] = \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We find that $x_3 = t$ is free and $x_1 = x_2 = x_3$, so

$$E_1 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

So $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis of the eigenspace corresponding to $\lambda = 1$. Check!

Finding a basis for E_2 is similar; see text.