# Math 1600 Lecture 27, Section 2, 10 Nov 2014

# Announcements:

Today we continue with 4.3. **Read** 4.3 and Appendix C for next class. Work through recommended homework questions.

**Tutorials:** Quiz 7 covers 4.2, the parts of Appendix D that we covered, and the part of 4.3 we finish today. No complex eigenvalues/roots.

**Office hour:** Monday, 3:00-3:30, MC103B.

**Help Centers:** Monday-Friday 2:30-6:30 in MC 106.

**Question:** If $P$ is invertible, how do $\det A$ and $\det(P^{-1}AP)$ compare?

They are equal:

$$\begin{aligned}\det(P^{-1}AP) &= \det(P^{-1})\det(A)\det(P) \\ &= \frac{1}{\det(P)}\det(A)\det(P) = \det A.\end{aligned}$$

## Partial review of last class: Section 4.3

**Definition:** If $A$ is $n \times n$, $\det(A - \lambda I)$ will be a degree $n$ polynomial in $\lambda$. It is called the **characteristic polynomial** of $A$, and $\det(A - \lambda I) = 0$ is called the **characteristic equation**.

By the fundamental theorem of invertible matrices, the solutions to the characteristic equation are exactly the eigenvalues.

**Finding eigenvalues and eigenspaces:** Let $A$ be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$.
2. Find the eigenvalues of $A$ by solving the characteristic equation $\det(A - \lambda I) = 0$.
3. For each eigenvalue $\lambda$, find a basis for the eigenspace

$E_\lambda = \mathrm{null}(A - \lambda I)$ by solving the system $(A - \lambda I)\vec{x} = \vec{0}$.

So we need to get good at solving polynomial equations. Solutions are called **zeros** or **roots**.

**Theorem D.4 (The Fundamental Theorem of Algebra):** A polynomial of degree $n$ has at most $n$ distinct roots.

Therefore:

**Theorem:** An $n \times n$ matrix $A$ has at most $n$ distinct eigenvalues.

Also:

**Theorem D.2 (The Factor Theorem):** Let $f$ be a polynomial and let $a$ be a constant. Then $a$ is a zero of $f(x)$ (i.e. $f(a) = 0$) if and only if $x - a$ is a factor of $f(x)$ (i.e. $f(x) = (x - a)g(x)$ for some polynomial $g$).

## New material: 4.3 continued

A root $a$ of a polynomial $f$ implies that $f(x) = (x - a)g(x)$. Sometimes, $a$ is also a root of $g(x)$, as we found above. Then $f(x) = (x - a)^2 h(x)$. The largest $k$ such that $(x - a)^k$ is a factor of $f$ is called the **multiplicity** of the root $a$ in $f$.

In the case of an eigenvalue, we call its multiplicity in the characteristic polynomial the **algebraic multiplicity** of this eigenvalue.

We also define the **geometric multiplicity** of an eigenvalue $\lambda$ to be the dimension of the corresponding eigenspace.

**Example 4.19:** Find the eigenvalues and eigenspaces of

$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$. Do partially, on board.

In this case, we find that $\lambda = 0$ has algebraic multiplicity 2 and geometric multiplicity 2.

These multiplicities will be important in Section 4.4.

**Theorem 4.15:** The eigenvalues of a triangular matrix are the entries on its main diagonal (repeated according to their algebraic multiplicity).

**Example:** If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 1 \end{bmatrix}$, then

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 0 \\ 4 & 5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda),$$

so the eigenvalues are $\lambda = 1$ (with algebraic multiplicity 2) and $\lambda = 3$ (with algebraic multiplicity 1).

**Question:** What are the eigenvalues of a diagonal matrix?

The eigenvalues are the diagonal entries.

**Question:** What are the eigenvalues of $\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$?

The characteristic polynomial is

$$\begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2),$$

so the eigenvalues are 2 and -2. Trick question.

**Question:** How can we tell whether a matrix $A$ is invertible using eigenvalues?

$A$ is invertible if and only if 0 is not an eigenvalue, because 0 being an eigenvalue is equivalent to $\mathrm{null}(A)$ being non-trivial, which is equivalent to $A$ not being invertible, by the fundamental theorem.

So we can extend the fundamental theorem with two new entries:

**Theorem 4.17:** Let $A$ be an $n \times n$ matrix. The following are equivalent:
a. $A$ is invertible.
b. $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.
c. $A\vec{x} = \vec{0}$ has only the trivial (zero) solution.
d. The reduced row echelon form of $A$ is $I_n$.
f. $\text{rank}(A) = n$
g. $\text{nullity}(A) = 0$
h. The columns of $A$ are linearly independent.
i. The columns of $A$ span $\mathbb{R}^n$.
j. The columns of $A$ are a basis for $\mathbb{R}^n$.
k. The rows of $A$ are linearly independent.
l. The rows of $A$ span $\mathbb{R}^n$.
m. The rows of $A$ are a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. $0$ is not an eigenvalue of $A$

## Eigenvalues of powers and inverses

Suppose $\vec{x}$ is an eigenvector of $A$ with eigenvalue $\lambda$. What can we say about $A^2$ or $A^3$? If $A$ is invertible, how about the eigenvalues/vectors of $A^{-1}$? On board.

We've shown:

**Theorem 4.18:** If $\vec{x}$ is an eigenvector of $A$ with eigenvalue $\lambda$, then $\vec{x}$ is an eigenvector of $A^k$ with eigenvalue $\lambda^k$. This holds for each integer $k \geq 0$, and also for $k < 0$ if $A$ is invertible.

In contrast to some other recent results, this one is very useful computationally:

**Example 4.21:** Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

**Solution:** By finding the eigenspaces of the matrix, we can show that

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Write $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Since $\vec{x} = 3\vec{v}_1 + 2\vec{v}_2$ we have

$$\begin{aligned} A^{10}\vec{x} &= A^{10}(3\vec{v}_1 + 2\vec{v}_2) = 3A^{10}\vec{v}_1 + 2A^{10}\vec{v}_2 \\ &= 3(-1)^{10}\vec{v}_1 + 2(2^{10})\vec{v}_2 = \begin{bmatrix} 3 + 2^{11} \\ -3 + 2^{12} \end{bmatrix} \end{aligned}$$

**Much faster** than repeated matrix multiplication, especially if $10$ is replaced with $100$.

This raises an interesting question. In the example, the eigenvectors were a basis for $\mathbb{R}^2$, so we could use this method to compute $A^k\vec{x}$ for any $\vec{x}$. However, last class we saw a $3 \times 3$ matrix with two one-dimensional eigenspaces, so the eigenvectors didn't span $\mathbb{R}^3$. We will study this further in Section 4.4, but right now we can answer a related question about linear independence.

**Theorem:** If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are eigenvectors of $A$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly independent.

**Proof in case $m = 2$:** If $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some $c$. Therefore

$$A\vec{v}_1 = A\,c\vec{v}_2 = cA\vec{v}_2$$

so

$$\lambda_1\vec{v}_1 = c\lambda_2\vec{v}_2 = \lambda_2\vec{v}_1$$

Since $\vec{v}_1 \neq \vec{0}$, this forces $\lambda_1 = \lambda_2$, a contradiction. $\square$

The general case is very similar; see text.

Next: how to become a Billionaire using the material from this course.