Math 1600 Lecture 29, Section 2, 14 Nov 2014

Announcements:

Today we finish 4.3 and start 4.4. Continue **reading** Section 4.4 for next class. Work through recommended homework questions.

Final exam: Monday, December 8, 9am to noon. See the course home page for final exam **conflict** policy. You should **immediately** notify the registrar or your Dean's office (and your instructor) of any conflicts! (Deadline Nov 21.)

Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Question: Can you find a nonzero complex number z such that $z^2 = 0$?

True/False: If z and w are complex numbers in the first quadrant, then so is zw.

Partial review of Appendix C

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is a symbol such that $i^2 = -1$.

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$, like vector addition.

Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

The **conjugate** of $z = a + bi$ is $\bar{z} = a - bi$. Reflection in real axis. We learned the properties of conjugation.

The **absolute value** or **modulus** of $z = a + bi$ is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}, \quad \text{the distance from the origin.}$$

Since $z\bar{z} = |z|^2$, we have that

$$\frac{z\bar{z}}{|z|^2} = 1 \quad \text{so} \quad \frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad (\text{for } z \neq 0)$$

This can be used to divide complex numbers:

$$\frac{w}{z} = \frac{w}{z}\,\frac{\bar{z}}{\bar{z}} = \frac{w\bar{z}}{|z|^2}\,.$$

We learned the properties of absolute value. One of them was $|wz| = |w||z|$.

A complex number $z = a + bi$ can also be expressed in **polar coordinates** (r, θ), where $r = |z| \geq 0$ and θ is such that

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

Then

$$z = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$$

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

So

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

(up to multiples of 2π).

In particular, if $z = r(\cos\theta + i\sin\theta)$, then $z^2 = r^2(\cos(2\theta) + i\sin(2\theta))$. It follows that the two **square roots** of z are

$$\pm\sqrt{r}(\cos(\theta/2) + i(\sin\theta/2))$$

Partial review of Section 4.3

The eigenvalues of a square matrix A can be computed as the **roots** (also called **zeros**) of the **characteristic polynomial**

$$\det(A - \lambda I)$$

If a is a root of a polynomial $f(x)$, then $f(x) = (x - a)g(x)$. Sometimes, a is also a root of $g(x)$. Then $f(x) = (x - a)^2 h(x)$. The largest k such that $(x - a)^k$ is a factor of f is called the **multiplicity** of the root a in f.

In the case of an eigenvalue, we call its multiplicity in the characteristic polynomial the **algebraic multiplicity** of this eigenvalue.

Example: Let $f(x) = x^2 - 2x + 1$. Since $f(1) = 1 - 2 + 1 = 0$, 1 is a root of f. And since $f(x) = (x - 1)^2$, 1 has multiplicity 2.

In the case of an eigenvalue, we call its multiplicity in the characteristic polynomial the **algebraic multiplicity** of this eigenvalue.

Newish material

Theorem D.4 (The Fundamental Theorem of Algebra): A polynomial of degree n has at most n distinct roots. In fact, the sum of the multiplicities is at most n.

Therefore:

Theorem: An $n \times n$ matrix A has at most n distinct eigenvalues. In fact, the sum of the algebraic multiplicities is at most n.

Complex eigenvalues and eigenvectors

This material isn't covered in detail in the text.

Example 4.7: Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (a) over $\mathbb{R}$ and (b) over $\mathbb{C}$.

Solution: We must solve

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1.$$

(a) Over $\mathbb{R}$, there are no solutions, so A has no real eigenvalues. This is why the Theorem above says "at most n". (This matrix represents rotation by 90 degrees, and we also saw geometrically that it has no real eigenvectors.)

(b) Over $\mathbb{C}$, the solutions are $\lambda = i$ and $\lambda = -i$. For example, the eigenvectors for $\lambda = i$ are the nonzero **complex** multiples of $\begin{bmatrix} i \\ 1 \end{bmatrix}$, since

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

In fact, $\lambda^2 + 1 = (\lambda - i)(\lambda + i)$, so each of these eigenvalues has algebraic multiplicity 1. So in this case the sum of the algebraic multiplicities is **exactly** 2.

The Fundamental Theorem of Algebra can be extended to say:

Theorem D.4 (The Fundamental Theorem of Algebra): A polynomial of degree n has at most n distinct **complex** roots. In fact, the sum of their multiplicities is **exactly** n.

Another way to put it is that over the complex numbers, every polynomial factors into **linear** factors.

Real matrices

Notice that i and $-i$ are complex conjugates of each other.

If the matrix A has only real entries, then the characteristic polynomial has real coefficients. Say it is

$$\det(A - \lambda I) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0,$$

with all of the a_i's real numbers. If z is an eigenvalue, then so is its complex conjugate $\bar{z}$, because

$$a_n \bar{z}^n + a_{n-1} \bar{z}^{n-1} + \cdots + a_1 \bar{z} + a_0$$
$$= \overline{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0} = \bar{0} = 0.$$

Theorem: The complex eigenvalues of a **real** matrix come in conjugate pairs.

Complex matrices

A complex matrix might have real or complex eigenvalues, and the complex eigenvalues do not have to come in conjugate pairs.

Examples: $\begin{bmatrix} 1 & 2 \\ 0 & i \end{bmatrix}$, $\begin{bmatrix} 1 & i \\ 0 & 2 \end{bmatrix}$.

General case

In general, don't forget that the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives the roots of $ax^2 + bx + c$, and these can be real (if $b^2 - 4ac \geqslant 0$) or complex (if $b^2 - 4ac < 0$). This formula also works if a, b and c are complex.

Also don't forget to try small integers first.

Example: Find the real and complex eigenvalues of $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

Solution:

$$\begin{vmatrix} 2-\lambda & 3 & 0 \\ 1 & 2-\lambda & 2 \\ 0 & -2 & 1-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 0 & 1-\lambda \end{vmatrix}$$
$$= (2-\lambda)(\lambda^2 - 3\lambda + 6) - 3(1-\lambda)$$
$$= -\lambda^3 + 5\lambda^2 - 9\lambda + 9.$$

By trial and error, $\lambda = 3$ is a root. So we factor:

$$-\lambda^3 + 5\lambda^2 - 9\lambda + 9 = (\lambda - 3)(-\lambda^2 + 2\lambda - 3)$$

We don't find any obvious roots for the quadratic factor, so we use the quadratic formula:

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(-1)(-3)}}{-2} = \frac{-2 \pm \sqrt{-8}}{-2}$$
$$= \frac{-2 \pm 2\sqrt{2}\,i}{-2} = 1 \pm \sqrt{2}\,i.$$

So the eigenvalues are 3, $1 + \sqrt{2}\,i$ and $1 - \sqrt{2}\,i$. The algebraic multiplicities are all 1, and $1 + 1 + 1 = 3$.

Note: Our questions always involve real eigenvalues and real eigenvectors unless we say otherwise. But there **will** be problems where we ask for complex eigenvalues.

Section 4.4: Similarity and Diagonalization

We're going to introduce a new concept that will turn out to be closely related to eigenvalues and eigenvectors.

Definition: Let A and B be $n \times n$ matrices. We say that A is **similar** to B if there is an invertible matrix P such that $P^{-1}AP = B$. When this is the case, we write $A \sim B$.

It is equivalent to say that $AP = PB$ or $A = PBP^{-1}$.

Example 4.22: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$. Then $A \sim B$, since

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}.$$

We also need to check that the matrix $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is invertible, which is the case since its determinant is 2.

It is tricky in general to find such a P when it exists. We'll learn a method that works in a certain situation in this section.