Math 1600 Lecture 6, Section 2, 17 Sep 2014

Announcements:

Read Sections 2.0 and 2.1 for next class. Work through recommended homework questions.

Quiz today and tomorrow in lab, covering 1.1, 1.2 and the code vectors part of 1.4.

Office hour: today, 11:30-noon, MC103B.

Help Centers: Monday-Friday 2:30-6:30 in MC 106 starting today (Sept 17).

Lecture notes (this page) available from course web page.

Partial review of last lecture:

Section 1.3: Lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

The **vector form** of the equation for a line ℓ is:

$$\vec{x} = \vec{p} + t\vec{d},$$

where $\vec{p}$ is the position vector of a chosen point on the line, $\vec{d}$ is a vector parallel to the line, and $t \in \mathbb{R}$.

If we expand the vector form into components, we get the **parametric form** of the equations for ℓ:

$$x = p_1 + td_1$$
$$y = p_2 + td_2$$
$$(z = p_3 + td_3 \quad \text{if we are in } \mathbb{R}^3)$$

Lines in $\mathbb{R}^2$

For a line in $\mathbb{R}^2$, there are additional ways to describe a line.

The **normal form** of the equation for ℓ is:

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p},$$

where $\vec{n}$ is a vector that is *normal = perpendicular* to ℓ.

If we write this out in components, with $\vec{n} = [a, b]$, we get the **general form** of the equation for ℓ:

$$ax + by = c,$$

where $c = \vec{n} \cdot \vec{p}$. When $b \neq 0$, this can be rewritten as $y = mx + k$, where $m = -a/b$ and $k = c/b$.

Note: All of these simplify when the line goes through the origin, as then you can take $\vec{p} = \vec{0}$.

Note: None of these equations is *unique*, as $\vec{p}$, $\vec{d}$ and $\vec{n}$ can all change. The general form is closest to being unique: it is unique up to an overall scale factor.

Lines in $\mathbb{R}^3$

There are also **normal** and **general forms** of equations for a line in $\mathbb{R}^3$, which I won't review here.

Planes in $\mathbb{R}^3$

Normal form:

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}.$$

When expanded into components, it gives the **general form**:

$$ax + by + cz = d,$$

where $\vec{n} = [a, b, c]$ and $d = \vec{n} \cdot \vec{p}$.

New material

Note: You can read off $\vec{n}$ from the general form. Two planes are parallel if and only if their normal vectors are parallel.

A plane can also be described in **vector form**. You need to specify a point $\vec{p}$ in the plane as well as two vectors $\vec{u}$ and $\vec{v}$ which are parallel to the plane but not parallel to each other.

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$$

When expanded into components, this gives the **parametric equations** for a plane:

$$\begin{aligned} x &= p_1 + su_1 + tv_1 \\ y &= p_2 + su_2 + tv_2 \\ z &= p_3 + su_3 + tv_3. \end{aligned}$$

Table 1.3 in the text summarizes this nicely (except for the one typo mentioned earlier).

It may seem like there are lots of different forms, but really there are two: vector and normal, and these can be expanded into components to give the parametric and general forms.

Example: Find all four forms of the equations for the plane in $\mathbb{R}^3$ which goes through the point $P = (1, 2, 0)$ and has normal vector

$\vec{n} = [2, 1, -1]$.

Solution: For $\vec{p} = [1, 2, 0]$ and $\vec{n} = [2, 1, -1]$, we have $\vec{n} \cdot \vec{p} = 4$. So the normal form is

$$\vec{n} \cdot \vec{x} = 4.$$

The general form is

$$2x + y - z = 4.$$

To get the vector form, we need two vectors parallel to the plane, so we need two vectors perpendicular to $\vec{n}$. Can get these by trial and error, for example, $\vec{u} = [-1, 2, 0]$ and $\vec{v} = [0, 1, 1]$. Then the vector form is

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}.$$

Expanding into components gives the parametric form:

$$\begin{aligned} x &= 1 - s \\ y &= 2 + 2s + t \\ z &= \phantom{2 + 2s + {}} t. \end{aligned}$$

You can also find parallel vectors by finding two other points Q and R in the plane and then taking $\vec{u} = \overrightarrow{PQ}$ and $\vec{v} = \overrightarrow{PR}$. If $\vec{u}$ and $\vec{v}$ are parallel, you need to try again.

True/false: The planes given by

$$2x + 3y + 4z = 7$$

and

$$4x + 6y + 8z = 9$$

are parallel.

True, because the normal vectors are $[2, 3, 4]$ and $[4, 6, 8]$, which are parallel.

If the 9 was changed to 14, the two planes would be **equal**, but the answer would still be **true**, as a plane is parallel to itself. The right hand side shifts the position of a plane, but not its orientation.

True/false: The lines given by

$$\vec{x} = \vec{p}_1 + t\vec{d}_1, \qquad \vec{p}_1 = [1,2,3], \qquad \vec{d}_1 = [2,0,-2]$$
$$\vec{x} = \vec{p}_2 + t\vec{d}_2, \qquad \vec{p}_2 = [2,4,6], \qquad \vec{d}_2 = [2,1,0]$$

are parallel.

False, because the direction vectors are $[2,0,-2]$ and $[2,1,0]$, which are not parallel. ($\vec{p}$ does not matter for this.)

Example: Find all four forms of the equations for the plane in $\mathbb{R}^3$ which goes through the points $P = (1,1,0)$, $Q = (0,1,2)$ and $R = (-1,2,1)$. Solution on board.

Cross products (Exploration after Section 1.3)

Given vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, we would like a way to produce a new vector that is orthogonal to both $\vec{u}$ and $\vec{v}$. The **cross product** does this.

Definition: The **cross product** of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} := [u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1].$$

Theorem: $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. That is, $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ and $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.

Explain on board.

Note: The cross product only makes sense in $\mathbb{R}^3$!

Can now finish the previous example, on board.

Theorem: The cross product also has the following properties:
(a) $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$!
(b) $\vec{u} \times \vec{0} = \vec{0}$
(c) $\vec{u} \times \vec{u} = \vec{0}$
(d) $\vec{u} \times k\vec{v} = k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v}$
(e) $\vec{u} \times k\vec{u} = \vec{0}$
(f) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

This is exercise 5 from the cross product exploration, and I encourage you to check these properties.

Distance from a point to a line (back to Section 1.3)

Recall the formula for **the projection of $\vec{v}$ onto $\vec{u}$**:

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right) \vec{u}.$$

Example: Find the distance from the point $B = (1, 3, 6)$ to the line through $P = (1, 1, 0)$ in the direction $\vec{d} = [0, -1, 1]$. Solution on board, leading to $d(B, \ell) = \|\vec{v} - \mathrm{proj}_{\vec{d}}(\vec{v})\| = 4\sqrt{2}$, where $\vec{v} = \overrightarrow{PB}$.

If the line was in $\mathbb{R}^2$ and had been described in normal form, one could instead compute $\|\mathrm{proj}_{\vec{n}}(\vec{v})\|$, which saves one step.