Math 1600 Lecture 7, Section 2, 19 Sep 2014

Announcements:

Read Section 2.2 for next class. Remember that the text gives more examples and explanation than I can give in a lecture. Work through recommended homework questions.

Office hour: Monday, 3:00-3:30, and Wednesday, 11:30-noon, MC103B.

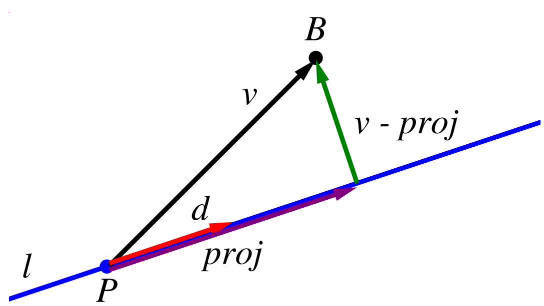
Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Review: Section 1.3: Distance from a point to a line

Recall the formula for **the projection of $\vec{v}$ onto $\vec{u}$**:

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}.$$

Example: Find the distance from the point $B = (1,3,6)$ to the line through $P = (1,1,0)$ in the direction $\vec{d} = [0,-1,1]$. We derived the formula $d(B,\ell) = \|\vec{v} - \mathrm{proj}_{\vec{d}}(\vec{v})\|$, where $\vec{v} = \overrightarrow{PB}$.



If the line was in $\mathbb{R}^2$ and had been described in normal form, one could instead compute $\|\mathrm{proj}_{\vec{n}}(\vec{v})\|$, which saves one step.

New material

Last bit of Section 1.3: Distance from a point to a plane

Example: Find the distance from the point $B = (1, 3, 6)$ to the plane $\mathcal{P}$ whose equation is $2x + y - z = 2$. Solution on board, leading to

$$d(B, \mathcal{P}) = \|\mathrm{proj}_{\vec{n}}(\vec{v})\| = \frac{|\vec{n} \cdot \vec{v}|}{\|\vec{n}\|} = |-3|/\sqrt{6} = 3/\sqrt{6}.$$

Section 2.1: Systems of Linear Equations

Definition: A **linear equation** in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

where the **coefficients** $a_1, \ldots, a_n$ and the **constant term** b are constants.

Linear equations:

$$2x - 5y = 10, \qquad r + \frac{1}{2}\, s = 0.5t - 2,$$

$$x = 0, \qquad x_1 - \sqrt{2}\, x_2 - (\sin \frac{\pi}{5})\, x_3 = 0.$$

Non-linear equations:

$$xy + z = 1, \quad x^2 + y^2 = 2, \quad \sin(x) = 0, \quad 2^y + z = 16.$$

A **solution** to $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$ is a vector $[s_1, \ldots, s_n]$ such that the equation is true when we substitute $x_1 = s_1, \ldots, x_n = s_n$. For example, $[10, 2]$ is a solution to $2x - 5y = 10$.

When a linear equation has two unknowns, its solutions form a line in $\mathbb{R}^2$. (The linear equation is the general form of this line.) To describe the solutions in parametric form, we can solve for one of the variables in terms of the other.

For example, for $2x - 5y = 10$, we can write $y = \frac{2}{5}\, x - 2$. If we set x to a parameter t, we get parametric solutions

$$x = t$$
$$y = \frac{2}{5}\,t - 2$$

or, more concisely, $[t, \frac{2}{5}\,t - 2]$.

When there are three variables, the solutions form a plane, and it can be described in parametric form by solving for one variable in terms of the other two.

The same works when there are n variables: we can solve for one in terms of all of the others, and get a solution with $n - 1$ parameters.

Systems of linear equations

Definition: A **system of linear equations** is a finite set of linear equations, each with the same variables. A **solution** to the system is a vector that satisfies *all* of the equations.

Example:

$$\begin{aligned} x + y &= 2 \\ -x + y &= 4 \end{aligned}$$

Is $[1, 1]$ a solution? How about $[-1, 3]$? How can we find all solutions? What's happening geometrically?

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Example:

$$x + y = 2$$
$$x + y = 3$$

Is $[1, 1]$ a solution? How about $[-1, 3]$? How can we find all solutions? What's happening geometrically?

A system is **consistent** if it has one or more solutions, and **inconsistent** if it has no solutions. We'll see later that a consistent system always has either one solution or infinitely many.

Solving a system

We started with the system on the left and produced the system on the right:

$$\begin{aligned} x + y &= 2, \\ -x + y &= 4, \end{aligned} \qquad \begin{aligned} x + \quad y &= 2 \\ 2y &= 6 \end{aligned}$$

The system on the right was **easy** to solve. These two systems are said to be **equivalent** because they have exactly the same solutions. (The geometry is different, though!)

Example 2.5: Similarly, a large system such as

$$\begin{aligned} x - y - \quad z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned}$$

is easy to solve, because of its **triangular** structure. The method is called **back substitution**:

$$\begin{aligned} z &= 2 \\ y &= 5 - 3z = 5 - 6 = -1 \\ x &= 2 + y + z = 2 - 1 + 2 = 3. \end{aligned}$$

So the unique solution is $[3, -1,\ 2]$.

Let's see how a general system can be converted into a system with a triangular form.

Example 2.6: We'll solve the system on the left

$$\begin{aligned} x - y - z &= 2 \\ 3x - 3y + 2z &= 16 \\ 2x - y + z &= 9 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

but to save time, we can write it as the **augmented matrix** on the right. Today, we'll show the equations as well.

To put it into triangular form, the first step is to eliminate the xs in equations 2 and 3. We write R_i for the ith equation or the ith row of the augmented matrix.

Replace R_2 with $R_2 - 3R_1$:

$$\begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ 2x - y + z &= 9 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

Replace R_3 with $R_3 - 2R_1$:

$$\begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ y + 3z &= 5 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

Now we can exchange rows 2 and 3, to end up in triangular form:

$$\begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

Hey! This is the system we solved earlier, so now we know that the solution is $[3, -1, 2]$.

This system and the original system have *exactly* the same solutions. (Explain.) We say they have the same **solution set** and therefore that they are **equivalent** systems.

Questions

True/false: The equation

$$\frac{x}{\sin(2)} + y = z$$

is linear.

True. Each variable appears to the first power, just multiplied by constants.

True/false: The system

$$\begin{aligned} 2x + 3y + 4z &= 7 \\ 4x + 6y + 8z &= 9 \end{aligned}$$

has no solutions.

True. Each equation describes a plane, and the planes are parallel. So they are either equal or don't intersect at all. Since the second equation is not a multiple of the first, they are not equal, so there are no points on both planes.

Alternatively, one can subtract twice the first equation from the second to get $0 = -5$, which has no solution.

True/false: The system

$$\begin{aligned} 2x + 3y + 4z &= 7 \\ 4x + 6y + 8z &= 14 \end{aligned}$$

has a unique solution.

False. Both equations describe the *same* plane, so there are infinitely many solutions.

Question: Solve the system

$$\begin{aligned} 2x + 3y &= 2 \\ x + 2y &= 2 \end{aligned}$$

geometrically and algebraically.

Geometrically: the two lines are not parallel (since the normal vectors [2,3] and [1,2] are not parallel), so they intersect in a single point. Sketch.

Algebraically: subtracting half of the first equation from the second gives

$$\begin{aligned} 2x + 3y &= 2 \\ \frac{1}{2}\,y &= 1 \end{aligned}$$

Now we apply back substitution. The second equation gives $y = 2$, and then the first equation gives $x = -2$. (Check that this satisfies the original equation!)

Question: How many solutions does the system

$$\begin{aligned} 2x + 3y &= 2 \\ x + 2y &= 2 \\ x + 4y &= 2 \end{aligned}$$

have?

This about this for next class.